Statistical properties of functionals of the paths of a particle diffusing in a one-dimensional random potential

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We present a formalism for obtaining the statistical properties of functionals and inverse functionals of the paths of a particle diffusing in a one-dimensional quenched random potential. We demonstrate the implementation of the formalism in two specific examples: (1) where the functional corresponds to the local time spent by the particle around the origin and (2) where the functional corresponds to the occupation time spent by the particle on the positive side of the origin, within an observation time window of size t. We compute the disorder average distributions of the local time, the inverse local time, the occupation time, and the inverse occupation time and show that in many cases disorder modifies the behavior drastically.

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I. INTRODUCTION

The statistical properties of functionals of a one-dimensional Brownian motion have been extensively studied and have found numerous applications in diverse fields ranging from probability theory [1–3], finance [4–6], mesoscopic physics [7], and computer science [8] and in understanding weather records [9]. The position $x(\tau)$ of a one-dimensional Brownian motion evolves with time τ via the Langevin equation

$$\frac{dx}{d\tau} = \eta(\tau), \tag{1}$$

starting from $x(0)=x_0$, where $\eta(\tau)$ is a thermal Gaussian white noise with mean $\langle \eta(\tau)\rangle=0$ and a correlator $\langle \eta(\tau)\eta(\tau')\rangle=\delta(\tau-\tau')$. A functional T is simply the integral up to time t:

$$T = \int_0^t V(x(\tau))d\tau, \tag{2}$$

where $V(x)$ is a prescribed non-negative function whose choice depends on the specific observable of interest. For a fixed initial position x_0 of the Brownian motion and a fixed observation time t, the value of T varies from one history or realization of the Brownian path $\{x(\tau)\}$ to another (see Fig. 1) and a natural question is, what is the probability density function (PDF) $P(T|t,x_0)$?

Following the path integral methods devised by Feynman [10], Kac showed [1,2] that the calculation of the PDF $P(T|t,x_0)$ can essentially be reduced to a quantum mechanics problem: namely, solving a single-particle Schrödinger equation in an external potential $V(x)$. This formalism is known in the literature as the celebrated Feynman-Kac formula. Subsequently, this method has been widely used to calculate the PDF of T with different choices of $V(x)$ as demanded by specific applications. This has been reviewed recently in Ref. [8]. In particular, the two most popular applications correspond, respectively, to the choices $V(x)=\delta(x-a)$ and $V(x)=\theta(x)$, where $\delta(x)$ is Dirac's delta function and $\theta(x)$ is the Heaviside step function. In the former case, the corresponding functional $T(a)=\int_0^t\delta(x(\tau)-a)d\tau$ has the following physical meaning: $T(a)da$ is just the time spent by the particle in the vicinity of the point a in space—i.e., in the region $[a,a+da]$—out of the total observation time t. Note that, by definition, $\int T(a)da=t$. The functional $T(a)$ is known as the "local time" (density) in the literature. In the second case $V(x)=\theta(x)$, the functional $T=\int_0^t\theta(x(\tau))d\tau$ measures the time spent by the particle on the positive side of the origin out of the total time t and is known as the "occupation" time. The probability distribution of the occupation time was originally computed by Lévy [11], $\int_0^T P(T'|t,0)dT'=\frac{2}{\pi}\arcsin(\sqrt{T/t})$, and is known as the arcsine law of Lévy. Since then, the local and occupation times for pure diffusion have been studied extensively by mathematicians [12–17]. Recently, the study of the occupation time has seen a revival in the physics literature and has been used in understanding the dynamics out of equilibrium in coarsening systems [18,19], ergodicity properties in anomalously diffusive processes [20,21], in renewal processes [22], in models related to spin glasses [23],

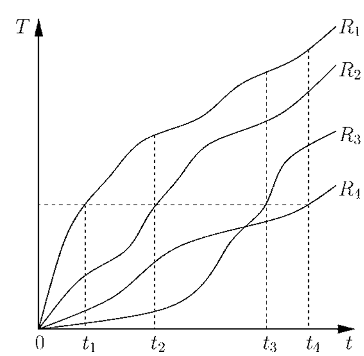

FIG. 1. Schematic plots of T defined by Eq. (2) as a function of t, corresponding to four different realizations of the paths [$\{x(\tau)\}$, for $0\leqslant\tau\leqslant t$] denoted by R_1, R_2, R_3, and R_4 respectively. For fixed t (t_1, t_2, t_3, or t_4, shown by vertical dashed lines), T takes different value for different realizations. On the other hand, for a fixed T (horizontal dashed line) the corresponding t is different for different realizations: t_1 for R_1, t_2 for R_2, t_3 for R_3, and t_4 for R_4.

in simple models of blinking quantum dots [24], and also in the context of persistence [25,26]. Local and occupation times have also been studied in the context of stochastic ergodicity breaking [27], first-passage time [28], diffusion-controlled reactions activated by catalytic sites [29], and diffusion on graphs [30,31]. In polymer science, a long flexible polymer of length t is often modeled by a Brownian path up to time t. In this context, the local time at a position $\vec{r}$ is proportional to the concentration of monomers at $\vec{r}$ in a polymer of length t.

A natural and important question is how to generalize the Feynman-Kac formalism to calculate the statistical properties of the functionals of the type in Eq. (2) when $x(\tau)$ is not just a pure diffusion process, but represents the position of a particle in an external random medium. While various properties of diffusion in random media have been widely studied in the past [32–35], the study of the statistical properties of functionals in random media is yet to receive its much deserved attention. In this paper we undertake this task. More precisely, we are interested in calculating the PDF $P(T|t,x_0)$ of a functional T as in Eq. (2) where $x(\tau)$ now evolves via the Langevin equation

$$\frac{dx}{d\tau} = F(x(\tau)) + \eta(\tau), \tag{3}$$

where $\eta(\tau)$ represents the thermal noise as in Eq. (1) and $F(x)=-dU/dx$ represents the external force, the derivative of the potential $U(x)$, felt by the particle. Most generally, the external potential consists of two parts, $U(x)=U_d(x)+U_r(x)$, a deterministic part $U_d(x)$ and a random part $U_r(x)$. The random part of the potential $U_r(x)$ is "quenched" in the sense that it does not change during the time evolution of the particle, but fluctuates from one sample to another according to some prescribed probability distribution. Consequently, the PDF $P(T|t,x_0)$ will also fluctuate from one sample of the random potential to another and the goal is to compute the disorder-averaged PDF $\overline{P(T|t,x_0)}$ where the overbar denotes the average over the distribution of the random potential. A popular model for the random potential is the celebrated Sinai model [36], where various disorder-averaged physical quantities can be computed analytically [32,33,37–42], and yet the results exhibit rich and nontrivial behaviors and also capture many of the qualitative behaviors of more complex realistic disordered systems. The Sinai model assumes that $U_r(x)=\sqrt{\sigma}B(x)$ where $B(x)$ represents a Brownian motion in space—i.e.,

$$\frac{dB}{dx} = \xi(x), \tag{4}$$

where $\xi(x)$ is a Gaussian white noise with mean $\langle\xi(x)\rangle=0$ and a correlator $\langle\xi(x)\xi(x')\rangle=\delta(x-x')$. The constant σ represents the strength of the random potential.

In this paper, we first present a generalization of the Feynman-Kac formalism to calculate the PDF $P(T|t,x_0)$ in the presence of an arbitrary external potential $U(x)$. To obtain explicit results using this formalism, we next assume that the random part of the potential is as in the Sinai model—i.e., that our external potential is of the form $U(x)=U_d(x)+\sqrt{\sigma}B(x)$, where $B(x)$ is a Brownian motion in space and $U_d(x)$ is the nonrandom deterministic part of the potential. It turns out that the asymptotic behavior of the disorder-averaged PDF $\overline{P(T|t,x_0)}$, quite generically, has three different qualitative behaviors depending on the curvature of the deterministic potential $U_d(x)$—i.e., whether $U_d(x)$ has a convex (concave-upward) shape representing a stable potential (i.e., attractive force towards the origin), a concave (concave-downward) shape representing unstable potential (a repulsive force away from the origin), or just flat indicating the absence of any external potential. To facilitate an explicit calculation, we model the deterministic potential simply by $U_d(x)=-\mu|x|$, so that $\mu<0$ represents a stable potential, $\mu>0$ represents an unstable potential, and $\mu=0$ represents a flat potential. This specific choice facilitates explicit calculation, but the results are qualitatively similar if one chooses another form of this potential. Thus, in our model, we will consider the external potential as

$$U(x) = -\mu|x| + \sqrt{\sigma}B(x), \tag{5}$$

where $B(x)=\int_0^x\xi(x')dx'$ is the trajectory of a Brownian motion in space (see Fig. 2). Note that the case $\mu=0$ corresponds to the pure Sinai model. Figure 2 shows typical realization of potentials for $\mu=0$, $\mu>0$, and $\mu<0$. The corresponding force in Eq. (3) is simply given by

$$F(x) = \mu\,\mathrm{sgn}(x) + \sqrt{\sigma}\xi(x). \tag{6}$$

We will demonstrate how to calculate explicitly, using our generalized Feynman-Kac formalism, the disorder-averaged PDF $\overline{P(T|t,x_0)}$ when the external potential is of the form given by Eq. (5). Despite the simplicity of the choice of the external potential, a variety of rich and interesting behaviors can be obtained by tuning the parameter μ/σ, as shown in this paper. We will present detailed results for the two functionals: namely, for the local time and the occupation time corresponding to the choices $V(x)=\delta(x)$ and $V(x)=\theta(x)$, respectively, in Eq. (2). Also, to keep the discussion simple, we will present our final results for $x_0=0$ corresponding to the particle starting at the origin. However, our method is not limited only to this specific choice. Some of these results were briefly announced in a previous Letter [43].

In addition, in this paper we also introduce the notion of "inverse functional," which is defined as follows. If $V(x)$ in Eq. (2) is non-negative, then for each path $\{x(\tau)\}$, T is a nondecreasing function of t, which we formally denote by $T=g(t|\{x(\tau)\},x_0)$. Therefore for a given realization of path $\{x(\tau)\}$ and given T there is a unique value of t (see Fig. 1), which we formally write as the inverse of the functional g,[1]

$$t = g^{-1}(T|\{x(\tau)\},x_0). \tag{7}$$

This inverse time t physically means the observation time that is required for any given path $\{x(\tau)\}$ in order to produce

[1]Strictly speaking this inverse does not exist always over a dense set of points. The inverse functional is properly defined in Ref. [46], p. 113, in the context of local time.

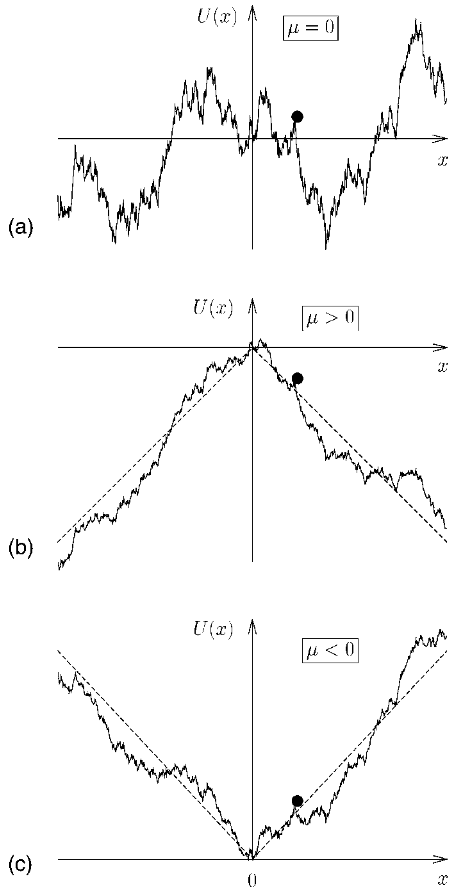

FIG. 2. A classical particle (represented by •) diffusing in a typical realization of the potential $U(x)=-\mu|x|+\sqrt{\sigma}B(x)$, where $B(x)$ represents the trajectory of a Brownian motion in space with $B(0)=0$. The three figures are for $\mu=0$, $\mu>0$, and $\mu<0$, respectively. The dashed lines show the potential for $\sigma=0$.

a prescribed value of T. Of course, for the same value T, for a different path $\{x(\tau)\}$, the value of t will be different. Thus, t is a random variable for a fixed T, which takes different values for different realizations of paths and we would like to compute its PDF, which we denote by $I(t|T,x_0)$ and by definition $\int_0^\infty I(t|T,0)dt=1$. Clearly, this PDF will also differ from sample to sample of the external potential in Eq. (5) and our goal is to obtain the disorder-averaged distribution $\overline{I(t|T,x_0)}$. In this paper, we present detailed results for $\overline{I(t|T,0)}$ again for the two choices of $V(x)=\delta(x)$ and $V(x)=\theta(x)$ corresponding to the local time and the occupation time, respectively. The inverse local and occupation times are useful for experimentalists as they provide an estimate of the required measurement time. For example, in the context of polymers, the inverse local time is the typical length of a polymer required to obtain a certain monomer concentration.

The rest of the paper is organized as follows. In Sec. II, we present our general approach for computing the PDF $P(T|t,x)$ of the functional T defined by Eq. (2) for a given t and the PDF $I(t|T,x)$ of the inverse functional defined by Eq. (7) for a given T, for a given sample of the random potential, for arbitrary starting position of the particle $x(0)=x$ and for arbitrary but non-negative $V(x)$. After this section we consider specific examples of local time and occupation time by setting $V(x)=\delta(x)$ and $V(x)=\theta(x)$, respectively. We will use different notations for the PDF's in the two examples to avoid any misunderstanding. In the first example, where T is the local time, we denote the PDF of the local time $P(T|t,0)$ for a given t by $P_{\mathrm{loc}}(T|t)$ and the PDF of the inverse local time $I(t|T,0)$ for a given T by $I_{\mathrm{loc}}(t|T)$. In the second example, where T is the occupation time we denote the PDF of the occupation time $P(T|t,0)$ for a given t by $P_{\mathrm{occ}}(T|t)$ and the PDF of the inverse occupation time $I(t|T,0)$ for a given T by $I_{\mathrm{occ}}(t|T)$. While our final goal is to obtain the disorder-averaged distributions $\overline{P_{\mathrm{loc}}(T|t)}$, $\overline{I_{\mathrm{loc}}(t|T)}$, $\overline{P_{\mathrm{occ}}(T|t)}$, and $\overline{I_{\mathrm{occ}}(t|T)}$, it is, however, instructive to study the pure case first before tackling the problem with disorder which is obviously harder. In the same spirit, we have presented detailed discussions of the local time, inverse local time, occupation time, and inverse occupation time for the pure case ($\sigma=0$) in Secs. III, V, VII, and IX, respectively, before computing their disorder average in Secs. IV, VI, VIII, and X, respectively. Section XI ontains some concluding remarks. Some of the details are relegated to the Appendixes. The results are summarized in Tables I–III.

II. GENERAL APPROACH

In this section we will show how to compute the PDF's $P(T|t,x)$ and $I(t|T,x)$ for arbitrary non-negative $V(x)$ and arbitrary starting position $x(0)=x$, for each realization of random force $F(x)$, by using a backward Fokker-Planck equation approach. In the following discussion we will denote the functional defined in Eq. (2) by $g(t|\{x(\tau)\},x_0)$, and use T as the value of the functional for a given path $[\{x(\tau)\}$, for $0\leq\tau\leq t]$.

Since $V(x)$ is considered to be non-negative, T defined by Eq. (2) has only positive support. Therefore, a natural step is to introduce the Laplace transform of the PDF $P(T|t,x)$ with respect to T:

$$Q_p(x,t)=\int_0^\infty P(T|t,x)e^{-pT}dT=\langle e^{-pg(t|\{x(\tau)\},x)}\rangle_{x(0)=x}$$
$$=\left\langle \exp\left\{-p\int_0^t V[x(t')]dt'\right\}\right\rangle_{x(0)=x}, \qquad (8)$$

where $\langle\cdots\rangle_{x(0)=x}$ denotes the average over all paths that start at the position $x(0)=x$ and propagate up to time t. Our aim is to derive a backward Fokker-Planck equation for $Q_p(x,t)$ with respect to the initial position $x(0)=x$.

We consider a particle starting from the initial position x and evolving via Eq. (3) up to time $t+\Delta t$. Then from Eq. (8), it follows that

$$Q_p(x,t+\Delta t)=\left\langle \exp\left\{-p\int_0^{t+\Delta t} V[x(t')]dt'\right\}\right\rangle_{x(0)=x}. \qquad (9)$$

Now we split the time interval $[0,t+\Delta t]$ into two parts: an infinitesimal interval $[0,\Delta t]$, over which the particle experi-

TABLE I. *Flat potential*. Disorder-averaged PDF's of the local, inverse local, occupation, and inverse occupation times of a particle starting at the origin, diffusing in the Sinai potential $U(x)=\sqrt{\sigma}B(x)$, where $B(x)$ represents the trajectory of a Brownian motion in space with the initial condition $B(0)=0$.

Pure case ($\sigma=0$)	Disordered case ($\sigma>0$)
$P_{\text{loc}}(T\|t)=\frac{\sqrt{2}}{\sqrt{\pi t}}\exp\left[-\frac{T^2}{2t}\right]$	$\overline{P_{\text{loc}}(T\|t)}\xrightarrow[T/t\ \text{fixed}]{t\to\infty,T\to\infty}\frac{1}{t\ln^2 t}f_1(T/t)$
	$f_1(y)=\frac{2}{y}e^{-y/\sigma}K_0(y/\sigma)$
$I_{\text{loc}}(t\|T)=\frac{T}{\sqrt{2\pi t^3}}\exp\left[-\frac{T^2}{2t}\right]$	$\overline{I_{\text{loc}}(t\|T)}\xrightarrow[t/T\ \text{fixed}]{t\to\infty,T\to\infty}\frac{1}{T\ln^2 T}g_1(t/T)$
	$g_1(x)=\frac{2}{x}e^{-1/\sigma x}K_0(1/\sigma x)$
$\overline{P_{\text{occ}}(T\|t)}=\frac{1}{\pi\sqrt{T(t-T)}}$,	$\overline{P_{\text{occ}}(T\|t)}=\overline{R_L(T\|t)}+\overline{R_L(t-T\|t)}$
$0<T<t$	$\overline{R_L(T\|t)}\xrightarrow{t\to\infty}\frac{1}{\ln t}R(T)$
	$R(T)\approx\frac{\sqrt{2}\sigma}{\sqrt{\pi T}}$, for small T
	$R(T)\sim\frac{1}{2T}$, for large T
$\overline{I_{\text{occ}}(t\|T)}=\frac{\sqrt{T}}{\pi t\sqrt{t-T}}\theta(t-T)$	$\overline{I_{\text{occ}}(t\|T)}\xrightarrow[t>T]{T\to\infty}\frac{1}{\ln T}I_3(t-T)$
	$I_3(\tau)\approx\frac{\sqrt{2}\sigma}{\sqrt{\pi\tau}}$, for small τ
	$I_3(\tau)\sim\frac{1}{2\tau}$, for large τ

ences an infinitesimal displacement Δx from its initial position x, and the remaining interval $[\Delta t, t+\Delta t]$, in which the particle evolves from a starting position $x+\Delta x$ (see Fig. 3). Since $x(0)=x$, from the first time interval $[0,\Delta t]$ one gets $\int_0^{\Delta t}V[x(t')]dt'=V(x)\Delta t+O[(\Delta t)^2]$. For the remaining interval $[\Delta t, t+\Delta t]$ the average $\langle\cdots\rangle_{x(0)=x}$ in Eq. (9) is now taken as follows: For a fixed Δx, the average is taken over all paths that start at position $x(\Delta t)=x+\Delta x$ and propagate up to time $t+\Delta t$, which gives $Q_p(x+\Delta x,t)$. Of course Δx is a random variable, since Δt is fixed. So now the average must be taken over all possible displacements Δx that can occur in Δt length of time. Therefore, from Eq. (9), we can now write down the evolution equation for $Q_p(x,t)$ as

$$Q_p(x,t+\Delta t)=e^{-pV(x)\Delta t}\langle Q_p(x+\Delta x,t)\rangle_{\Delta x}, \qquad (10)$$

where $\langle\cdots\rangle_{\Delta x}$ denotes the average with respect to all possible displacements Δx. Now assuming small Δx, using Taylor expansion one gets

$$\langle Q_p(x+\Delta x,t)\rangle_{\Delta x}=Q_p(x,t)+\langle\Delta x\rangle\frac{\partial Q_p}{\partial x}+\frac{1}{2}\langle(\Delta x)^2\rangle\frac{\partial^2 Q_p}{\partial x^2}+\cdots\ . \qquad (11)$$

In the limit $\Delta t\to 0$, integrating Eq. (3) one gets

$$\Delta x=F(x)\Delta t+\int_0^{\Delta t}\eta(\tau)d\tau+O[(\Delta t)^2]. \qquad (12)$$

Using the zero mean and the uncorrelated properties of the noise we get

$$\lim_{\Delta t\to 0}\frac{\langle\Delta x\rangle}{\Delta t}=F(x),\quad \lim_{\Delta t\to 0}\frac{\langle(\Delta x)^2\rangle}{\Delta t}=1. \qquad (13)$$

Therefore, using Eq. (11) and $e^{-pV(x)\Delta t}=1-pV(x)\Delta t+\cdots$ in

TABLE II. *Unstable potential*. Disorder-averaged PDF's of the local, inverse local, occupation, and inverse occupation times of a particle starting at the origin, diffusing in the unstable random potential $U(x)=-\mu|x|+\sqrt{\sigma}B(x)$, where $\mu>0$ and $B(x)$ represents the trajectory of a Brownian motion in space with the initial condition $B(0)=0$. We denote $\nu=\mu/\sigma$.

Pure case ($\sigma=0$)	Disordered case ($\sigma>0$)
$P_{\text{loc}}(T\vert t)\xrightarrow{t\to\infty}P_{\text{loc}}(T)$,	$\overline{P_{\text{loc}}(T\vert t)}\xrightarrow{t\to\infty}\overline{P_{\text{loc}}(T)}$,
$P_{\text{loc}}(T)=2\mu e^{-2\mu T}$	$\overline{P_{\text{loc}}(T)}=2\mu(1+\sigma T)^{-(2\nu+1)}$
$I_{\text{loc}}(t\vert T)=\frac{T}{\sqrt{2\pi t^3}}\exp\left[-\frac{(T+\mu t)^2}{2t}\right]$	$\overline{I_{\text{loc}}(t\vert T)}\xrightarrow[t/T\ \text{fixed}]{t\to\infty,T\to\infty}\frac{1}{T^{2\nu+1}}g_2(t/T)$
$+(1-\exp[-2\mu T])\delta(t-\infty)$	$+(1-[1+\sigma T]^{-2\nu})\delta(t-\infty)$
	$g_2(x)=\left[\frac{2\sqrt{\pi}}{\sigma^{2\nu-1}\Gamma^2(\nu)}\right]\frac{e^{-2/\sigma x}}{(\sigma x)^{2\nu+1}}U(1/2,1+\nu,2/\sigma x)$
$P_{\text{occ}}(T\vert t)=R_L(T\vert t)+R_L(t-T\vert t)$	$\overline{P_{\text{occ}}(T\vert t)}=\overline{R_L(T\vert t)}+\overline{R_L(t-T\vert t)}$
$R_L(T\vert t)\xrightarrow{t\to\infty}R_L(T)$	$\overline{R_L(T\vert t)}\xrightarrow{t\to\infty}\overline{R_L(T)}$
$R_L(T)=\mu\sqrt{2}\exp\left[-\frac{\mu^2 T}{2}\right]$	$\overline{R_L(T)}\approx\frac{\mu\sqrt{2}}{\sqrt{\pi T}}$, for small T
$\times\left[\frac{1}{\sqrt{\pi T}}-\frac{3\mu}{\sqrt{2}}\exp\left(\frac{9\mu^2}{2}T\right)\text{erfc}\left(\frac{3\mu}{\sqrt{2}}\sqrt{T}\right)\right]$	$\overline{R_L(T)}\sim e^{-bT}$, for large T b is given by the zero of $K_\nu(\sqrt{2p/\sigma})$ closest to origin in the left part of the complex-p plane.
$R_L(T)\approx\frac{\mu\sqrt{2}}{\sqrt{\pi T}}$, for small T	
$R_L(T)\approx\frac{\sqrt{2}}{9\mu\sqrt{\pi}}\frac{e^{-\mu^2 T/2}}{T^{3/2}}$, for large T	
$I_{\text{occ}}(t\vert T)\xrightarrow[t>T]{T\to\infty}I_1(t-T)+\frac{1}{2}\delta(t-\infty)$	$\overline{I_{\text{occ}}(t\vert T)}\xrightarrow[t>T]{T\to\infty}I_4(t-T)+\frac{1}{2}\delta(t-\infty)$
$I_1(\tau)=\mu\sqrt{2}\exp\left[-\frac{\mu^2\tau}{2}\right]$	$I_4(\tau)\approx\frac{\mu\sqrt{2}}{\sqrt{\pi\tau}}$, for small τ
$\times\left[\frac{1}{\sqrt{\pi\tau}}-\frac{3\mu}{\sqrt{2}}\exp\left(\frac{9\mu^2}{2}\tau\right)\text{erfc}\left(\frac{3\mu}{\sqrt{2}}\sqrt{\tau}\right)\right]$	$I_4(\tau)\sim e^{-b\tau}$, for large τ b is the same constant as above.
$I_1(\tau)\approx\frac{\mu\sqrt{2}}{\sqrt{\pi\tau}}$, for small τ	
$I_1(\tau)\approx\frac{\sqrt{2}}{9\mu\sqrt{\pi}}\frac{e^{-\mu^2\tau/2}}{\tau^{3/2}}$, for large τ	

Eq. (10), then dividing both sides by Δt, and taking the limit $\Delta t\to 0$, one arrives at the "backward" Fokker-Planck equation

$$\frac{\partial Q_p}{\partial t}=\frac{1}{2}\frac{\partial^2 Q_p}{\partial x^2}+F(x)\frac{\partial Q_p}{\partial x}-pV(x)Q_p, \tag{14}$$

with the initial condition $Q_p(x,0)=1$, which is easily checked by Eq. (8). The advantage of the above equation over the usual Feynman-Kac formalism [1,2] is that, in the latter case one has a "forward" Fokker-Planck equation (spatial derivative with respect to the final position), where after

TABLE III. *Stable potential*. Disorder-averaged PDF's of the local, inverse local, occupation, and inverse occupation times of a particle starting at the origin, diffusing in the stable random potential $U(x)=-\mu|x|+\sqrt{\sigma}B(x)$, where $\mu<0$ and $B(x)$ represents the trajectory of a Brownian motion in space with the initial condition $B(0)=0$. We denote $\nu=|\mu|/\sigma$.

Pure case ($\sigma=0$)	Disordered case ($\sigma>0$)
$P_{\rm loc}(T\|t)\sim\exp\left[-t\Phi\left(\frac{T}{t}\right)\right]$, $\begin{cases}t\to\infty, T\to\infty\\ T/t \text{ fixed}\end{cases}$	$\overline{P_{\rm loc}(T\|t)}\xrightarrow[T/t\text{ fixed}]{t\to\infty,T\to\infty}\frac{1}{t}f_2(T/t)$
$\Phi(r)=\frac{1}{2}(r-\|\mu\|)^2$, near $r=\|\mu\|$	$f_2(y)=\left[\frac{2\sqrt{\pi}}{\sigma\Gamma^2(\nu)}\right]\left(\frac{y}{\sigma}\right)^{2\nu-1}e^{-2y/\sigma}U(1/2,1+\nu,2y/\sigma)$
$I_{\rm loc}(t\|T)=\frac{T}{\sqrt{2\pi t^3}}\exp\left[-\frac{(T-\|\mu\|t)^2}{2t}\right]$	$\overline{I_{\rm loc}(t\|T)}\xrightarrow[t/T\text{ fixed}]{t\to\infty,T\to\infty}\frac{1}{T}g_3(t/T)$
	$g_3(x)=\left[\frac{2\sigma\sqrt{\pi}}{\Gamma^2(\nu)}\right]\frac{e^{-2/\sigma x}}{(\sigma x)^{2\nu+1}}U(1/2,1+\nu,2/\sigma x)$
$P_{\rm occ}(T\|t)\sim\exp\left[-t\Phi\left(\frac{T}{t}\right)\right]$, $\begin{cases}t\to\infty, T\to\infty\\ T/t \text{ fixed}\end{cases}$	$\overline{P_{\rm occ}(T\|t)}\xrightarrow[T/t\text{ fixed}]{t\to\infty,T\to\infty}\frac{1}{t}f_{\rm o}(T/t)$
$\Phi(r)=2\mu^2\left(r-\frac{1}{2}\right)^2$, near $r=\frac{1}{2}$	$f_{\rm o}(y)=\frac{1}{B(\nu,\nu)}[y(1-y)]^{\nu-1}$, $0\leq y\leq 1$
$I_{\rm occ}(t\|T)=I_2(t-T,T)\theta(t-T)$	$\overline{I_{\rm occ}(t\|T)}\xrightarrow[t/T\text{ fixed}]{t\to\infty,T\to\infty}\frac{1}{T}g_0(t/T)$
$I_2(\tau,T)\xrightarrow{\text{large } T}\frac{\|\mu\|T}{\sqrt{2\pi\tau^3}}\exp\left[-\frac{\mu^2(\tau-T)^2}{2\tau}\right]$	$g_{\rm o}(x)=\frac{1}{B(\nu,\nu)}\frac{(x-1)^{\nu-1}}{x^{2\nu}}$, $x>1$

obtaining the solution of the differential equation, one has to again perform an additional step of integration over the final position. In contrast, Eq. (14) involves the spatial derivatives with respect to the initial position of the particle, and hence no additional step of integration over the final position is required.

The standard practice of attacking the partial differential equations of above type is by using the method of Laplace transformation. We define the Laplace transform of $Q_p(x,t)$ with respect to t:

$$u(x)=\int_0^\infty Q_p(x,t)e^{-\alpha t}dt=\int_0^\infty dte^{-\alpha t}\int_0^\infty dTe^{-pT}P(T|t,x), \tag{15}$$

where for notational convenience, we have suppressed the α and p dependence of $u(x)$. Now by taking Laplace transform of Eq. (14) with respect to t we obtain the ordinary differential equation

$$\frac{1}{2}u''(x)+F(x)u'(x)-[\alpha+pV(x)]u(x)=-1, \tag{16}$$

where $u'(x)=du/dx$. The appropriate boundary conditions $u(x\to\pm\infty)$ are to be derived from the observation that if the particle starts at $x\to\pm\infty$ it will never cross the origin in finite time. Note that Eq. (16) is valid for each sample of the quenched random force $F(x)$. Thus, in principle, from the solution $u(x)$ one obtains $P(T|t,x)$ by inverting the double Laplace transform in Eq. (15) for each sample of quenched random potential and then takes the average over the disorder.

Our next goal in this section is to show how to compute the PDF $I(t|T,x)$ for a given sample of the quenched random force $F(x)$. It turns out that $I(t|T,x)$ is related to the PDF $P(T|t,x)$ in their Laplace space as shown below. By definition we have

$$I(t|T,x)=\langle\delta(t-g^{-1}(T|\{x(\tau)\},x))\rangle_x. \tag{17}$$

However, it is elementary that for each realization of path $\{x(\tau)\}$

$$\delta(t-g^{-1}(T|\{x(\tau)\},x))=\delta(T-g(t|\{x(\tau)\},x))\left|\frac{dT}{dt}\right|, \tag{18}$$

where $|dT/dt|$ is the usual Jacobian of the transformation, which is simply dT/dt as both T and t have only positive support. It immediately follows from the above two equations that

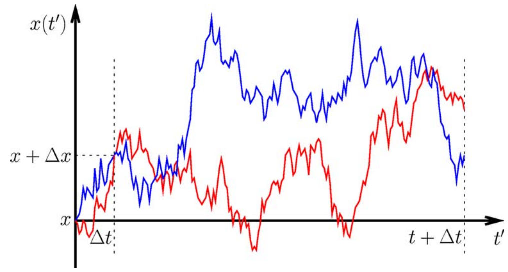

FIG. 3. (Color online) Schematic plot showing two realizations of positions of a particle starting from the initial position $x(0)=x$ evolve to a position $x+\Delta x$ in time Δt and then starting from the position $x+\Delta x$ evolve up to time $t+\Delta t$ (color online).

$$I(t|T,x) = \left\langle \delta(T - g(t|\{x(\tau)\},x))\frac{dT}{dt}\right\rangle_x. \tag{19}$$

Therefore, Laplace transform of $I(t|T,x)$ with respect to T reads

$$\int_0^\infty dTe^{-pT}I(t|T,x) = \left\langle e^{-pg(t|\{x(\tau)\},x)}\frac{dg}{dt}\right\rangle_x = -\frac{1}{p}\frac{\partial}{\partial t}Q_p(x,t), \tag{20}$$

where $Q_p(x,t)$ is given by Eq. (8). Now taking a further Laplace transform in Eq. (20) with respect to t, it is straightforward to obtain

$$\int_0^\infty dte^{-\alpha t}\int_0^\infty dTe^{-pT}I(t|T,x) = \frac{1-\alpha u(x)}{p}. \tag{21}$$

Thus, we have established, via Eqs. (20) and (21), the relationships between the Laplace transforms of the PDF of the functional T defined by Eq. (2) and the PDF of the inverse functional defined by Eq. (7). Hence, again in principle, from the solution $u(x)$ of the ordinary differential equation (16), one obtains $I(t|T,x)$ by inverting the double Laplace transform in Eq. (21) for each sample of quenched random potential, and then takes the average over the disorder. Note that putting $\alpha=0$ in Eq. (21) and inverting the Laplace transform with respect to p immediately verifies the normalization condition $\int_0^\infty I(t|T,x)dt=1$.

In the rest of the paper, we will demonstrate how to implement this formalism for the particular examples of the local time corresponding to the choice $V(x)=\delta(x)$ and the occupation time corresponding to the choice $V(x)=\theta(x)$. Since in these examples we consider the starting position of the particle to be the origin, we need to only find the solution $u(0)$ of the differential equation (16). In each example, we will consider the pure cases ($\sigma=0$) first, which help us anticipate the general features of the results in the disordered case ($\sigma>0$) studied later.

III. LOCAL TIME WITHOUT DISORDER (σ=0)

In this case $V(x)=\delta(x)$ corresponds to T in Eq. (2) being the local time in the vicinity of the origin and $P(T|t,0)$ in Eq. (15) being $P_{\mathrm{loc}}(T|t)$—the PDF of the local time T for a given observation time window of size t and the starting position of the particle $x(0)=0$. For our purpose we only need the solution $u(0)$ of the differential equation (16), which corresponds to the starting position of the particle being the origin. However, to obtain $u(0)$ we have to solve Eq. (16) in the entire region of x with the boundary conditions $u(x\to\pm\infty)$ which are derived from the following observation. If the initial position $x\to\pm\infty$, the particle cannot reach the origin in finite time, which means that the local time $T=0$. Therefore, by substituting $P(T|t,x\to\pm\infty)\to\delta(T)$ in Eq. (15) one obtains the boundary conditions

$$u(x\to\pm\infty) = \frac{1}{\alpha}. \tag{22}$$

We have to obtain the solutions $u(x)=u_+(x)$ for $x>0$ and $u(x)=u_-(x)$ for $x<0$ by solving Eq. (16) separately in the respective two regions,

$$\frac{1}{2}u''_\pm(x) + F(x)u'_\pm(x) - \alpha u_\pm(x) = -1, \tag{23}$$

with the boundary conditions $u_+(x\to\infty)=1/\alpha$ and $u_-(x\to-\infty)=1/\alpha$, and then matching the two solutions $u_+(x)$ and $u_-(x)$ at $x=0$. The matching conditions are

$$u_+(0) = u_-(0) = u(0), \quad u'_+(0) - u'_-(0) = 2pu(0). \tag{24}$$

The first condition follows from the fact that the solution must be continuous at $x=0$, and the second one is derived by integrating Eq. (16) across $x=0$.

By making a constant shift $u_\pm(x)=1/\alpha+A_\pm y_\pm(x)$, from Eq. (23) one finds that $y_\pm(x)$ satisfy the p-independent homogeneous equation

$$\frac{1}{2}y''_\pm(x) + F(x)y'_\pm(x) - \alpha y_\pm(x) = 0, \tag{25}$$

with the boundary conditions $y_+(x\to\infty)\to0$ and $y_-(x\to-\infty)\to0$. The constants $A_\pm$ are determined by the matching conditions given in Eq. (24), which can be rewritten as

$$A_+y_+(0) = A_-y_-(0) = u(0) - \frac{1}{\alpha}, \tag{26a}$$

$$A_+y'_+(0) - A_-y'_-(0) = 2pu(0). \tag{26b}$$

Eliminating the constants $A_\pm$ from Eq. (26), we obtain the Laplace transform $u(0)$, defined by Eq. (15) with $P(T|t,0)\equiv P_{\mathrm{loc}}(T|t)$, as

$$u(0) = \int_0^\infty dte^{-\alpha t}\int_0^\infty dTe^{-pT}P_{\mathrm{loc}}(T|t) = \frac{\lambda(\alpha)}{\alpha[p+\lambda(\alpha)]}, \tag{27}$$

where $\lambda(\alpha)$ is simply given by

$$\lambda(\alpha) = \frac{z_-(0) - z_+(0)}{2} \quad \text{with } z_\pm(x) = \frac{y'_\pm(x)}{y_\pm(x)}. \qquad (28)$$

Note that putting $p=0$ in Eq. (27) and then inverting the Laplace transform with respect to α readily verifies the normalization condition

$$\int_0^\infty P_{\rm loc}(T|t)dT = 1. \qquad (29)$$

Since $\lambda(\alpha)$ is independent of p, inverting the Laplace transform in Eq. (27) with respect to p yields

$$G(\alpha) = \int_0^\infty dt e^{-\alpha t} P_{\rm loc}(T|t) = \frac{\lambda(\alpha)}{\alpha} \exp[-\lambda(\alpha)T], \quad (30)$$

which is valid for any arbitrary force $F(x)$. In the following subsections we will consider qualitatively different types of deterministic potentials to derive more explicit results.

A. Flat potential

We first consider the simple Brownian motion without any external force, $F(x)=0$. In this case the solutions of Eq. (25) are obtained as

$$y_\pm(x) = y_\pm(0)\exp[\mp x\sqrt{2\alpha}]. \qquad (31)$$

Using the solutions in Eq. (28) one gets $\lambda(\alpha)=\sqrt{2\alpha}$ and hence the Laplace transform $G(\alpha)$ in Eq. (30) becomes

$$G(\alpha) = \int_0^\infty dt e^{-\alpha t} P_{\rm loc}(T|t) = \frac{\sqrt{2}}{\sqrt{\alpha}} e^{-\sqrt{2\alpha}T}. \qquad (32)$$

Now inverting the Laplace transform with respect to α, one finds that the distribution of the local time is Gaussian for all T and t,

$$P_{\rm loc}(T|t) = \frac{\sqrt{2}}{\sqrt{\pi t}} \exp\left[-\frac{T^2}{2t}\right]. \qquad (33)$$

B. Unstable potential

Now we consider the case of a Brownian particle moving in an unstable potential $U(x)$ such that $U(x\to\pm\infty)\to-\infty$. The corresponding repulsive force $F(x)$ drives the particle eventually either to $+\infty$ or to $-\infty$. The PDF of the local time, $P_{\rm loc}(T|t)$, in the case of an unstable potential tends to a steady distribution $P_{\rm loc}(T)$ as $t\to\infty$, which can be computed explicitly. To see this consider the function $G(\alpha)$ in Eq. (30). By making a change of variable $\tau=\alpha t$, it follows from Eq. (30) that

$$G(\alpha) = \frac{1}{\alpha}\int_0^\infty d\tau P_{\rm loc}\left(T\Big|\frac{\tau}{\alpha}\right). \qquad (34)$$

Assuming $P_{\rm loc}(T|t\to\infty)=P_{\rm loc}(T)$, we find from the above equation that $G(\alpha)\to P_{\rm loc}(T)/\alpha$ as $\alpha\to 0$. Comparing this behavior with Eq. (30) gives

$$P_{\rm loc}(T) = \lambda(0)\exp[-\lambda(0)T], \qquad (35)$$

provided $\lambda(0)$ is a finite positive number. Thus generically, for all repulsive force $F(x)$, the local time distribution has a universal Poisson distribution in the limit $t\to\infty$. The only dependence on the precise form of the force $F(x)$ is through the rate constant $\lambda(0)$.

The rate constant $\lambda(0)$ can be expressed in terms of the force $F(x)$ in a more explicit manner. Putting $\alpha=0$ in Eq. (25) and solving the resulting equation with the boundary conditions $y_+(x\to\infty)\to 0$ and $y_-(x\to-\infty)\to 0$ we get

$$y_+(x) = y_+(0)\frac{\int_x^\infty \psi^2(y)dy}{\int_0^\infty \psi^2(y)dy}, \quad x>0, \qquad (36)$$

$$y_-(x) = y_-(0)\frac{\int_{-\infty}^x \psi^2(y)dy}{\int_{-\infty}^0 \psi^2(y)dy}, \quad x<0, \qquad (37)$$

where $\psi(y)=\exp[-\int_0^y F(x)dx]$. Substituting these results into Eq. (28) gives the rate constant as

$$\lambda(0) = \frac{1}{2}\left[\frac{1}{\int_{-\infty}^0 \psi^2(y)dy} + \frac{1}{\int_0^\infty \psi^2(y)dy}\right]. \qquad (38)$$

Let us now consider a simple example where the potential $U(x)=-\mu|x|$ with $\mu>0$, corresponding to the repulsive force $F(x)=\mu\,\mathrm{sgn}(x)$ from the origin. In this case $\psi(y)=\exp[-\mu|y|]$ and hence from Eq. (38) we get $\lambda(0)=2\mu$.

C. Stable potential

We now turn our attention to the complementary situation when the potential $U(x)$ is stable—i.e., $U(x\to\pm\infty)\to\infty$. In this case the force $F(x)$ is attractive towards the origin so that the system eventually reaches a well-defined stationary state. The stationary probability distribution $p(x)$ for the position of the particle is given by the Gibbs measure

$$p(x) = \frac{e^{-2U(x)}}{Z}, \qquad (39)$$

where $U(x)=-\int_0^x F(x')dx'$ and Z is the partition function,

$$Z = \int_{-\infty}^\infty e^{-2U(x)}dx. \qquad (40)$$

In this case the Laplace transform $G(\alpha)$ of the PDF of the local time $P_{\rm loc}(T|t)$ is still given by Eq. (30). However, unlike the unstable potential in the previous section, the distribution $P_{\rm loc}(T|t)$ does not approach a steady state as $t\to\infty$. Instead it has a rather different asymptotic behavior.

To deduce this asymptotic behavior, let us first consider the average local time $\langle T\rangle=\int_0^t\langle\delta[x(t')]\rangle dt'$. For large t', the average $\langle \delta[x(t')] \rangle$ approaches its stationary value $\langle \delta[x(t')] \rangle \to p(0)$, where $p(0)=1/Z$ from Eq. (39). Hence as $t\to\infty$ the ratio T/t approaches the limit

$$\frac{\langle T \rangle}{t} \to \frac{1}{Z}, \tag{41}$$

where Z is given by Eq. (40). Thus, for large t, the average local time scales linearly with time t, which indicates that the natural scaling limit in this case is when $t\to\infty, T\to\infty$ but keeping the ratio $r=T/t$ fixed. We will see that in this scaling limit the local time distribution $P_{\rm loc}(T|t)$ tends to the following asymptotic form:

$$P_{\rm loc}(T|t) \sim \exp\left[-t\Phi\left(\frac{T}{t}\right)\right], \tag{42}$$

where $\Phi(r)$ is a large deviation function.

To compute the large deviation function we first substitute this presumed asymptotic form of $P_{\rm loc}(T|t)$ given by Eq. (42) in the Laplace transform $G(\alpha)=\int_0^\infty e^{-\alpha t} P_{\rm loc}(T|t)dt$ and then make a change of variable $r=T/t$ in the integration. The resulting integral can be evaluated in the large-T limit by the method of steepest descent, which gives $G(\alpha)\sim\exp[-TW(\alpha)]$ where $W(\alpha)=\min_r[\{\alpha+\Phi(r)\}/r]$. Comparing this result with Eq. (30) gives

$$\min_r\left[\frac{\alpha+\Phi(r)}{r}\right] = \lambda(\alpha), \tag{43}$$

where $\lambda(\alpha)$ is given by Eq. (28). Thus $\lambda(\alpha)$ is just the Legendre transform of $\Phi(r)$. Inversion of this transform gives the exact large deviation function

$$\Phi(r) = \max_\alpha[-\alpha + r\lambda(\alpha)], \tag{44}$$

with $\lambda(\alpha)$ given by Eq. (28). This is a general result valid for any confining potential $U(x)$.

We will now explicitly compute the large deviation $\Phi(r)$ for the particular potential given by Eq. (5) with $\mu<0$ and $\sigma=0$. Substituting the corresponding force $F(x)=-|\mu|\mathrm{sgn}(x)$ in Eq. (25) and solving the resulting differential equations with the boundary conditions $y_+(x\to\infty)\to 0$ and $y_-(x\to-\infty)\to 0$ we get

$$y_\pm(x) = y_\pm(0)\exp[\mp(-|\mu| + \sqrt{\mu^2+2\alpha})x]. \tag{45}$$

Substituting these results in Eq. (28) we get $\lambda(\alpha)=-|\mu|+\sqrt{\mu^2+2\alpha}$. From Eq. (44) one then gets the large deviation function

$$\Phi(r) = \frac{1}{2}(r-|\mu|)^2. \tag{46}$$

It turns out that for this particular form of the force $F(x)=-|\mu|\mathrm{sgn}(x)$, the Laplace transform in Eq. (30) can be inverted to get the local time distribution $P_{\rm loc}(T|t)$ exactly for all T and t. The calculations are presented in Appendix A. We find that in the large-t limit, the distribution reduces to the asymptotic form

$$P_{\rm loc}(T|t) \approx \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{t}{2}\left(\frac{T}{t}-|\mu|\right)^2\right], \tag{47}$$

near the mean $\langle T\rangle=|\mu|t$, which verifies the result obtained above by the large deviation function calculation.

In fact, the limiting Gaussian form of the distribution of the local time near its mean value is quite generic for any stable potentials (where the system eventually becomes ergodic) and is just the manifestation of the central limit theorem. From the definition, $T-\langle T\rangle=\int_0^t\{\delta[x(t')]-\langle\delta[x(t')]\rangle\}dt'$, it follows that when $T\to\langle T\rangle$, the random variables $\delta[x(t')]-\langle\delta[x(t')]\rangle$ at different times t' become only weakly correlated. Then, in the limit when t is much larger than the correlation time between these variables, one expects the central limit theorem to hold which predicts a Gaussian form for T near its mean value $\langle T\rangle$.

IV. LOCAL TIME WITH DISORDER ($\sigma>0$)

So far we have considered the case where the random part of the potential was not present. In this section we will study the effect of the randomness by adding a random part to the potential. In particular, we will consider the diffusive motion of the particle when the force $F(x)$ is given by Eq. (6) with $\sigma>0$.

Equation (30) still remains valid for each realization of the force $F(x)$—i.e., for each realization of $\{\xi(x)\}$. Our aim is to compute the average of the PDF of the local time $\overline{P_{\rm loc}(T|t)}$ over the noise history $\{\xi(x)\}$. From Eq. (30), one needs to know the distribution of $\lambda(\alpha)=[z_-(0)-z_+(0)]/2$, which is now a random variable since $F(x)$ is random. The variables $-z_+(0)$ and $z_-(0)$ are independent of each other, and therefore their joint probability distribution factorizes to the individual distributions. The calculations of these distributions are presented in Appendix B. Using the distributions of $z_\pm(0)$ from Eqs. (B9) and (B11), respectively, with $a_\pm=\alpha$, one gets

$$\overline{\exp[-\lambda(\alpha)T]} = \left[\frac{q(T)}{q(0)}\right]^2, \tag{48}$$

with

$$q(T) = \int_0^\infty w^{\mu/\sigma-1}\exp\left[-\frac{1}{2\sigma}\left\{w(1+\sigma T)+\frac{2\alpha}{w}\right\}\right]dw \tag{49}$$

$$=2(2\alpha)^{\mu/2\sigma}(1+\sigma T)^{-\mu/2\sigma}K_{\mu/\sigma}\left(\frac{\sqrt{2\alpha(1+\sigma T)}}{\sigma}\right), \tag{50}$$

where $K_\nu(x)$ is the modified Bessel function of order ν [44] and $K_{-\nu}(x)=K_\nu(x)$. Averaging Eq. (30) over disorder we finally get the exact formula

$$\int_0^\infty dt e^{-\alpha t}\overline{P_{\rm loc}(T|t)} = -\frac{1}{\alpha q^2(0)}\frac{d}{dT}[q^2(T)]. \tag{51}$$

However, it is not an easy task to invert the Laplace transform to get the exact distribution $\overline{P_{\rm loc}(T|t)}$ for all T and t. In the following subsections we will extract the asymptotic behaviors of $\overline{P_{\text{loc}}(T|t)}$, for the three cases, when the deterministic part of the potential is (i) flat corresponding to $\mu=0$, (ii) unstable corresponding to $\mu>0$, and (iii) stable corresponding to $\mu<0$.

A. Flat potential ($\mu=0$): Sinai model

We first consider a particle diffusing in the continuous Sinai potential—i.e., $\mu=0$ in Eq. (5). Our aim is to find out how this random potential modifies the behavior of the local time. In this case substituting $q(T)$ and $q(0)$ from Eq. (50) with $\mu=0$ in Eq. (51) we get the Laplace transform of the disorder-averaged local time distribution as

$$\int_0^\infty dt e^{-\alpha t}\overline{P_{\text{loc}}(T|t)} = -\frac{1}{\alpha K_0^2(\sqrt{2\alpha}/\sigma)}\frac{\partial}{\partial T}K_0^2\left(\frac{\sqrt{2\alpha(1+\sigma T)}}{\sigma}\right). \tag{52}$$

We will now consider the interesting limit where both t and T are large, but the ratio $y=T/t$ is kept fixed. This corresponds to taking the limit $\alpha\to 0$ with $\alpha T=s$ keeping fixed. In this limit,

$$K_0\left(\frac{\sqrt{2\alpha}}{\sigma}\right)\to -\frac{1}{2}\ln\alpha \tag{53}$$

and

$$K_0\left(\frac{\sqrt{2\alpha(1+\sigma T)}}{\sigma}\right)\to K_0\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right). \tag{54}$$

Therefore, substituting $t=s/\alpha y$ and $T=s/\alpha$ in Eq. (52), in the limit $\alpha\to 0$ we get

$$\int_0^\infty dy e^{-s/y}\left[\frac{s}{\alpha y^2}\overline{P_{\text{loc}}\left(\frac{s}{\alpha}\Big|\frac{s}{\alpha y}\right)}\right] = -\frac{4}{\ln^2\alpha}\frac{\partial}{\partial s}K_0^2\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right). \tag{55}$$

The above equation suggests that, in the limit $t\to\infty$ and $T\to\infty$, while their ratio T/t is kept fixed, $\overline{P_{\text{loc}}(T|t)}$ should have the scaling form

$$\overline{P_{\text{loc}}(T|t)} = \frac{1}{t\ln^2 t}f_1(T/t). \tag{56}$$

Now substituting the above form in Eq. (55) and making the change of variable $\tilde{y}=1/y$, we obtain, after straightforward simplification,

$$\int_0^\infty d\tilde{y}e^{-s\tilde{y}}\left[\frac{f_1(1/\tilde{y})}{\tilde{y}^2}\right] = 4K_0^2\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right). \tag{57}$$

Note that the right-hand side of the above equation is simply the Laplace transform of the function $f_1(1/\tilde{y})/\tilde{y}^2$. Therefore by using the identity

$$\int_0^\infty \frac{e^{-a^2/4\omega}}{\omega}e^{-s\omega}d\omega = 2K_0(a\sqrt{s}) \tag{58}$$

and the convolution property of Laplace transform, we can invert the Laplace transform in Eq. (57) with respect to s.

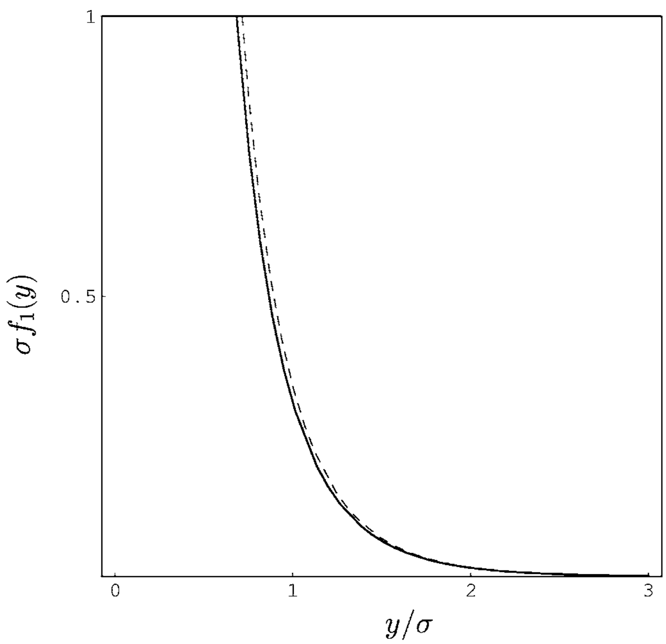

FIG. 4. The scaling function $f_1(y)$ in Eq. (56). The solid line is plotted by using Eq. (62), and the dashed line is plotted by using the limiting form $f_1(y)\sim\sqrt{2\pi\sigma}y^{-3/2}e^{-2y/\sigma}$ as $y\to\infty$.

Inverting the Laplace transform and after simplification we finally get

$$f_1(1/\tilde{y}) = 2\tilde{y}\int_0^{1/2}\frac{dx}{x(1-x)}\exp\left[-\frac{1}{2\sigma\tilde{y}x(1-x)}\right]. \tag{59}$$

Therefore the scaling function $f_1(y)$ is simply given by

$$f_1(y) = \frac{2}{y}\int_0^{1/2}\frac{dx}{x(1-x)}\exp\left[-\frac{y}{2\sigma x(1-x)}\right]. \tag{60}$$

By substituting $x(1-x)=1/z$ gives

$$f_1(y) = \frac{2}{y}\int_4^\infty\frac{dz}{\sqrt{z(z-4)}}\exp\left(-\frac{y}{2\sigma}z\right), \tag{61}$$

where the integral can be evaluated exactly [44], which finally gives the scaling function in Eq. (56) as

$$f_1(y) = \frac{2}{y}e^{-y/\sigma}K_0(y/\sigma). \tag{62}$$

However, the scaling given by Eq. (56) breaks down for very small y (very small T) when $y\ll\sigma$. The scaling function is displayed in Fig. 4. In the large-y limit, using the asymptotic the behavior $K_\nu(x)\sim\sqrt{\pi/2x}e^{-x}$ from Eq. (62) we find that $f_1(y)\sim\sqrt{2\pi\sigma}y^{-3/2}e^{-2y/\sigma}$.

B. Unstable potential ($\mu>0$)

In this case the behavior in the limit $t\to\infty$ can be obtained by either setting $\alpha=0$ in the integral in Eq. (49) or taking the $\alpha\to 0$ limit in $K_\nu(\cdot)$ in Eq. (50), which gives

$$q(T) \to \Gamma(\mu/\sigma)(2\sigma)^{\mu/\sigma}(1+\sigma T)^{-\mu/\sigma}, \tag{63}$$

where $\Gamma(x)$ is the gamma function [45]. Substituting $q(T)$ and $q(0)$ in Eq. (51) and inverting the Laplace transform with respect to α gives

$$\overline{P_{\mathrm{loc}}(T|t)} = 2\mu(1+\sigma T)^{-2\mu/\sigma-1}; \tag{64}$$

i.e., in the limit $t\to\infty$, the distribution $\overline{P_{\mathrm{loc}}(T|t)}$ tends to a steady-state distribution $\overline{P_{\mathrm{loc}}(T)}$ for all $T \geqslant 0$. The disorder-averaged local time distribution has a broad power-law distribution even though for each sample the local time has a narrow exponential distribution [see Eq. (35) in Sec. III B]. This indicates wide sample-to-sample fluctuations and lack of self-averaging.

C. Stable potential ($\mu < 0$)

In this case substituting $q(T)$ and $q(0)$ from Eq. (50) in Eq. (51) and denoting $\nu = |\mu|/\sigma$ we get

$$\int_0^\infty dt e^{-\alpha t}\overline{P_{\mathrm{loc}}(T|t)} = -\frac{1}{\alpha K_\nu^2(\sqrt{2\alpha}/\sigma)} \times \frac{\partial}{\partial T}\left[(1+\sigma T)^{\nu/2}K_\nu\left(\frac{\sqrt{2\alpha(1+\sigma T)}}{\sigma}\right)\right]^2. \tag{65}$$

We consider the scaling limit where both t and T are large, but their ratio $y=T/t$ is kept fixed. This corresponds to taking the limit $\alpha\to 0$ with keeping $\alpha T = s$ fixed, which gives the following limiting forms:

$$(1+\sigma T) \to \frac{\sigma s}{\alpha}, \tag{66}$$

$$K_\nu\left(\frac{\sqrt{2\alpha}}{\sigma}\right) \to \frac{\Gamma(\nu)}{2}\left(\frac{\sigma\sqrt{2}}{\sqrt{\alpha}}\right)^\nu, \tag{67}$$

$$K_\nu\left(\frac{\sqrt{2\alpha(1+\sigma T)}}{\sigma}\right) \to K_\nu\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right). \tag{68}$$

Substituting the above limits on the right-hand side of Eq. (65) and making a change of variables $t=s/\alpha y$ and $T=s/\alpha$ on the left-hand side, it is straightforward to get

$$\int_0^\infty dy e^{-s/y}\left[\frac{s}{\alpha y^2}\overline{P_{\mathrm{loc}}\left(\frac{s}{\alpha}\middle|\frac{s}{\alpha y}\right)}\right] = -\frac{4}{(2\sigma)^\nu\Gamma^2(\nu)}\frac{\partial}{\partial s}\left[s^{\nu/2}K_\nu\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right)\right]^2, \tag{69}$$

in the limit $\alpha\to 0$. This suggests the limiting form for $\overline{P_{\mathrm{loc}}(T|t)}$,

$$\overline{P_{\mathrm{loc}}(T|t)} \to \frac{1}{t}f_2(T/t), \tag{70}$$

in the scaling limit $t\to\infty$ and $T\to\infty$ with a fixed ratio $y=T/t$. To compute the scaling function we substitute the above scaling form in Eq. (69) and make the change of variable $\tilde{y}=1/y$. Then Eq. (69) simplifies to the Laplace transform

$$\int_0^\infty d\tilde{y}e^{-s\tilde{y}}\left[\frac{f_2(1/\tilde{y})}{\tilde{y}^2}\right] = \frac{4}{(2\sigma)^\nu\Gamma^2(\nu)}\left[s^{\nu/2}K_\nu\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right)\right]^2, \tag{71}$$

which can be inverted with respect to s, by using the identity

$$\int_0^\infty\left(\frac{a}{2}\right)^\nu\frac{e^{-a^2/4\omega}}{\omega^{\nu+1}}e^{-s\omega}d\omega = 2s^{\nu/2}K_\nu(a\sqrt{s}) \tag{72}$$

and the convolution property of the Laplace transform. After simplification, the inverse Laplace transform gives

$$f_2(y) = \frac{2y^{2\nu-1}}{(2\sigma)^{2\nu}\Gamma^2(\nu)}\int_0^{1/2}\frac{dx}{x^{\nu+1}(1-x)^{\nu+1}} \times\exp\left[-\frac{y}{2\sigma x(1-x)}\right]. \tag{73}$$

By making a change of variable $x(1-x)=1/z$ in the above integral, it can be presented in the form

$$f_2(y) = \frac{2y^{2\nu-1}}{(2\sigma)^{2\nu}\Gamma^2(\nu)}\int_4^\infty z^{\nu-1/2}(z-4)^{-1/2} \times\exp\left(-\frac{y}{2\sigma}z\right)dz, \tag{74}$$

which now can be expressed in more elegant form [45] as

$$f_2(y) = \left[\frac{2\sqrt{\pi}}{\sigma\Gamma^2(\nu)}\right]\left(\frac{y}{\sigma}\right)^{2\nu-1}e^{-2y/\sigma}U(1/2,1+\nu,2y/\sigma), \tag{75}$$

where $U(a,b,x)$ is the confluent hypergeometric function of the second kind (also known as Kummer's function of the second kind) [45], which has the following limiting behaviors:

$$U(1/2,1+\nu,x) \approx \frac{\Gamma(\nu)}{\sqrt{\pi}}x^{-\nu} \quad \text{for small } x, \tag{76a}$$

$$U(1/2,1+\nu,x) \sim \frac{1}{\sqrt{x}} \quad \text{for large } x. \tag{76b}$$

The scaling function $f_2(y)$ is displayed in Fig. 5. Using the limiting behaviors from Eq. (76), one finds that the scaling function decays as $f_2(y)\sim y^{(4\nu-3)/2}e^{-2y/\sigma}$ for large y. For small y, the scaling function behaves as $f_2(y)\sim y^{\nu-1}$, which increases with y for $\nu>1$, but which, however, diverges when $y\to 0$ for $\nu<1$, a behavior qualitatively similar to the Sinai case (see Fig. 4). For $\nu<1$, the disorder wins over the strength of the stable potential. In that situation when the particle gets trapped in the wells of the random potential, the weak external deterministic force often cannot lift it out of the well and send toward the origin. Therefore, the scaling

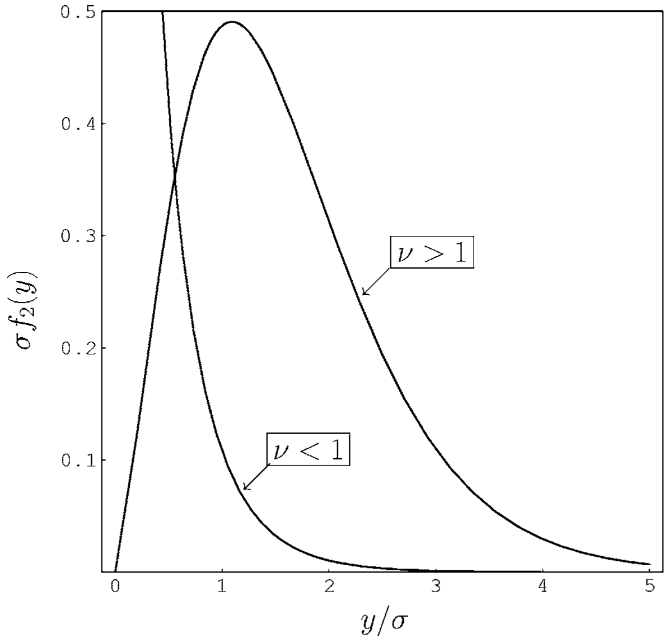

FIG. 5. The scaling function $f_2(y)$ in Eq. (70), plotted by using Eq. (75). $\nu = |\mu|/\sigma$.

function $f_2(y)$ carries very large weight near $y=0$ (which corresponds to very small local time T for a given observation time t).

Note that, for the particular value $\nu=1/2$, the scaling function has a simple form $f_2(y)=\sqrt{2/\pi\sigma y}\exp(-2y/\sigma)$.

V. INVERSE LOCAL TIME WITHOUT DISORDER (σ=0)

The inverse local time means how long one has to observe the particle until the total time spent in the infinitesimal neighborhood of the origin is T. The double Laplace transform of the PDF of the inverse local time is obtained by simply putting $x=0$ in Eq. (21). The corresponding $u(0)$ in Eq. (21), which is nothing but the double Laplace transform of the PDF of local time, has already been evaluated in Sec. III and is given by Eq. (27). Substituting $u(0)$ and replacing $I(t|T,0)$ with the PDF of the inverse local time $I_{\mathrm{loc}}(t|T)$, after straightforward simplification, for $x=0$, Eq. (21) reads

$$\int_0^\infty dt e^{-\alpha t}\int_0^\infty dT e^{-pT} I_{\mathrm{loc}}(t|T) = \frac{1}{p+\lambda(\alpha)}, \tag{77}$$

where $\lambda(\alpha)$ is given by Eq. (28), which depends on the force $F(x)$ through Eq. (25). Inverting the Laplace transform with respect to p gives the general formula

$$\int_0^\infty dt e^{-\alpha t} I_{\mathrm{loc}}(t|T) = \exp[-\lambda(\alpha)T], \tag{78}$$

valid for arbitrary force $F(x)$, a result known in the mathematics literature [17,46].

We first consider the pure case where the force given by Eq. (6) with $\sigma=0$. Substituting solutions of Eq. (25) for

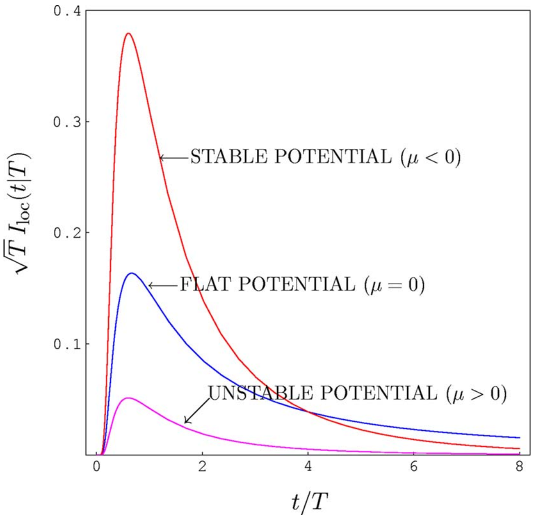

FIG. 6. (Color online) The PDF's of the inverse local time for stable ($\mu=-1/2$), flat ($\mu=0$), and unstable ($\mu=1/2$) potentials, plotted using Eq. (80) and $T=2$.

$F(x)=\mu\,\mathrm{sgn}(x)$ in Eq. (28) we obtain $\lambda(\alpha)=\mu+\sqrt{\mu^2+2\alpha}$. Now using this $\lambda(\alpha)$ in Eq. (78) and making a change of the parameter $\alpha=\beta-\mu^2/2$ we get

$$\int_0^\infty dt e^{-\beta t}[e^{\mu^2 t/2} I_{\mathrm{loc}}(t|T)] = e^{-\mu T} e^{-\sqrt{2\beta}T}, \tag{79}$$

where the right-hand side is simply the Laplace transform of $e^{\mu^2 t/2} I_{\mathrm{loc}}(t|T)$ with respect to t. The Laplace transform can be inverted to obtain the exact PDF of the inverse local time,

$$I_{\mathrm{loc}}(t|T) = \frac{T}{\sqrt{2\pi t^3}}\exp\left[-\frac{(T+\mu t)^2}{2t}\right], \tag{80}$$

with the normalization condition

$$\int_0^\infty I_{\mathrm{loc}}(t|T)dt = e^{-(\mu+|\mu|)T} = \begin{cases} 1 & \text{for } \mu \leqslant 0, \\ e^{-2\mu T} & \text{for } \mu > 0, \end{cases} \tag{81}$$

which is simply obtained by putting $\alpha=0$ in Eq. (78). As we infer from Eq. (80), although in the limit $t\to 0$ the inverse local time distribution $I_{\mathrm{loc}}(t|T)\sim\exp(-T^2/2t)$ is independent of μ, for large t it depends on the nature of the potential, as shown in Fig. 6. While in the absence of any force—i.e., $\mu=0$—the inverse local time distribution has a power-law tail $I_{\mathrm{loc}}(t|T)\sim t^{-3/2}$, for the stable potential—i.e., $\mu<0$; it decays exponentially $I_{\mathrm{loc}}(t|T)\sim\exp(-\mu^2 t/2)$. On the other hand, when the potential is unstable, $\mu>0$, as we see from Eq. (81), the distribution $I_{\mathrm{loc}}(t|T)$ is not normalized to unity. In this case the particle escapes to $\pm\infty$ with probability $(1-e^{-2\mu T})$ and Eq. (80) gives the distribution only for those events where the particle does not escape to $\pm\infty$. Therefore, for $\mu>0$, it is appropriate to represent the full normalized distribution as

$$I_{\rm loc}(t|T) = \frac{T}{\sqrt{2\pi t^3}}\exp\left[-\frac{(T+\mu t)^2}{2t}\right] + (1-e^{-2\mu T})\,\delta(t-\infty). \tag{82}$$

Note that the second term does not show up in the Laplace transform of $I_{\rm loc}(t|T)$ with respect to t.

VI. INVERSE LOCAL TIME WITH DISORDER ($\sigma > 0$)

In this section, we switch on the disorder by considering $\sigma > 0$ in the force given by Eq. (6). In the presence of disorder, taking the disorder average of Eq. (78) gives

$$\int_0^\infty dt e^{-\alpha t}\overline{I_{\rm loc}(t|T)} = \overline{\exp[-\lambda(\alpha)T]}, \tag{83}$$

with $\lambda(\alpha)=[z_-(0)-z_+(0)]/2$, where $-z_+(0)$ and $z_-(0)$ are independent random variables, whose distributions are given by Eqs. (B9) and (B11), respectively, with $a_\pm = \alpha$. The object $\overline{\exp[-\lambda(\alpha)T]}$ on the right-hand side of Eq. (83) has already been evaluated in Sec. IV, which is given by Eq. (48). In the following subsections we will determine the behavior of $\overline{I_{\rm loc}(t|T)}$ in the scaling limit $t\to\infty$, $T\to\infty$, while keeping their ratio $x=t/T$ fixed, for the three qualitatively different cases: (i) $\mu=0$, (ii) $\mu>0$, and (iii) $\mu<0$.

A. Flat potential ($\mu=0$): Sinai model

Following the analysis of Sec. IV A, in the limit $\alpha\to 0$ with keeping $\alpha T=s$ fixed,

$$\overline{\exp[-\lambda(\alpha)T]} \to \frac{4}{\ln^2\alpha}K_0^2\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right). \tag{84}$$

Therefore, substituting $t=xs/\alpha$ and $T=s/\alpha$, in the limit $\alpha\to 0$, Eq. (83) reads

$$\int_0^\infty dx e^{-sx}\left[\frac{s}{\alpha}\overline{I_{\rm loc}\left(\frac{sx}{\alpha}\Big|\frac{s}{\alpha}\right)}\right] = \frac{4}{\ln^2\alpha}K_0^2\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right). \tag{85}$$

This suggests the scaling form

$$\overline{I_{\rm loc}(t|T)} = \frac{1}{T\ln^2 T}g_1(t/T) \tag{86}$$

in the limit $t\to\infty$, $T\to\infty$ but keeping $x=t/T$ fixed. Substituting this scaling form in Eq. (85), after straightforward simplification one obtains

$$\int_0^\infty dx e^{-sx}g_1(x) = 4K_0^2\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right). \tag{87}$$

Now direct comparison of the above equation with Eq. (57) gives $g_1(x)=f_1(1/x)/x^2$, where $f_1(x)$ is given by Eq. (62). Substituting $f_1(1/x)$ one obtains the scaling function $g_1(x)$ as

$$g_1(x) = \frac{2}{x}e^{-1/\sigma x}K_0(1/\sigma x), \tag{88}$$

which is displayed in Fig. 7. The scaling function increases as $g_1(x)\approx\sqrt{2\pi\sigma}x^{-1/2}\exp(-2/\sigma x)$ for small x and decays as $g_1(x)\sim 2\ln(\sigma x)/x$ at large x.

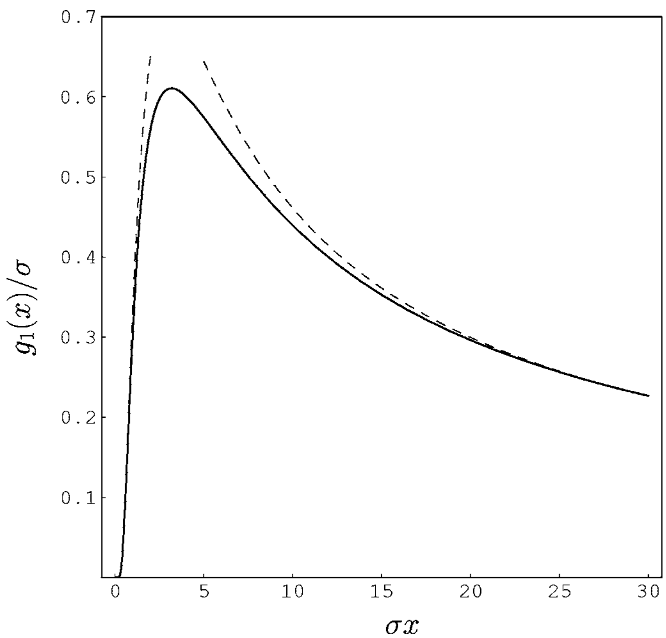

FIG. 7. The scaling function $g_1(y)$ in Eq. (86). The solid line is plotted by using Eq. (88), and the dashed line is plotted by using the limiting forms $g_1(x)\approx\sqrt{2\pi\sigma}x^{-1/2}\exp(-2/\sigma x)$ for small x and $g_1(x)\sim 2\ln(\sigma x)/x$ for large x.

B. Unstable potential ($\mu>0$)

In this case the right-hand side of Eq. (83) is given by

$$\overline{\exp[-\lambda(\alpha)T]} = (1+\sigma T)^{-\nu}\frac{K_\nu^2(\sqrt{2\alpha(1+\sigma T)}/\sigma)}{K_\nu^2(\sqrt{2\alpha}/\sigma)}, \tag{89}$$

with $\nu=\mu/\sigma$. Putting $\alpha=0$ in the above equation gives the normalization condition $\int_0^\infty \overline{I_{\rm loc}(t|T)}dt=(1+\sigma T)^{-2\nu}$, which implies that for the unstable potential, where the force is repulsive from the origin, the particle escapes to $\pm\infty$ with probability $1-(1+\sigma T)^{-2\nu}$ and the disorder-averaged PDF $\overline{I_{\rm loc}(t|T)}$ obtained by inverting the Laplace transform in Eq. (83) represents only those events where the particle does not escape to $\pm\infty$.

Now in the limit of $\alpha\to 0$ with $\alpha T=s$ keeping fixed, one gets

$$K_\nu\left(\frac{\sqrt{2\alpha}}{\sigma}\right) \to \frac{\Gamma(\nu)}{2}T^{\nu/2}\left(\frac{\sigma\sqrt{2}}{\sqrt{s}}\right)^\nu, \tag{90}$$

$$K_\nu\left(\frac{\sqrt{2\alpha(1+\sigma T)}}{\sigma}\right) \to K_\nu\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right). \tag{91}$$

Therefore Eq. (89) becomes

$$\overline{\exp[-\lambda(\alpha)T]} \to \frac{4T^{-2\nu}}{(2\sigma^3)^\nu\Gamma^2(\nu)}\left[s^{\nu/2}K_\nu\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right)\right]^2. \tag{92}$$

In the corresponding limit $T\to\infty$, $t\to\infty$, but keeping their ratio $x=t/T$ fixed, using the scaling form

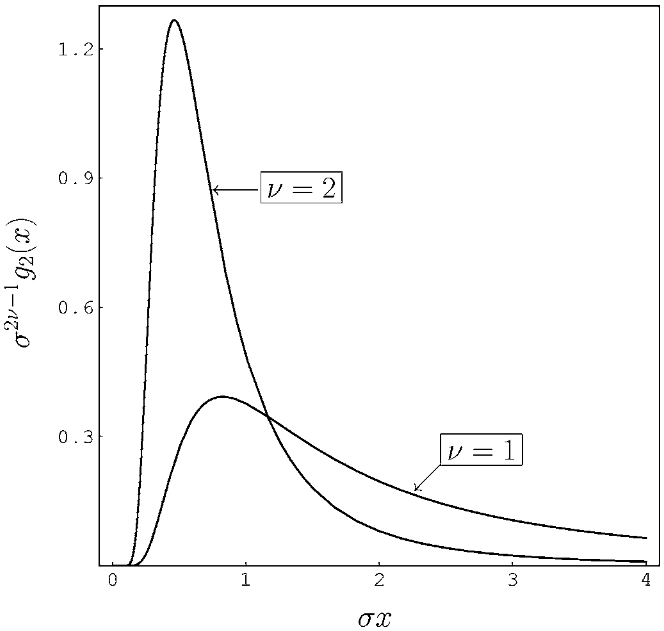

FIG. 8. The scaling functions $g_2(x)$ in Eq. (93) plotted by using Eq. (95). $\nu=|\mu|/\sigma$.

$$\overline{I_{\rm loc}(t|T)} = \frac{1}{T^{2\nu+1}} g_2(t/T) \tag{93}$$

in Eq. (83) one finally arrives at the Laplace transform

$$\int_0^\infty e^{-sx} g_2(x) dx = \frac{4}{(2\sigma^3)^\nu \Gamma^2(\nu)} \left[s^{\nu/2} K_\nu\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right)\right]^2. \tag{94}$$

The Laplace transform can be inverted with respect to s to obtain the scaling function $g_2(x)$, and in fact the inversion has already been done in Sec. IV C. Comparing the above equation with Eq. (71) readily gives $g_2(x)=\sigma^{-2\nu} f_2(1/x)/x^2$ where $f_2(x)$ is given by Eq. (75). Substituting $f_2(1/x)$ gives

$$g_2(x) = \left[\frac{2\sigma\sqrt{\pi}}{\sigma^{2\nu}\Gamma^2(\nu)}\right] \frac{e^{-2/\sigma x}}{(\sigma x)^{2\nu+1}} U(1/2, 1+\nu, 2/\sigma x), \tag{95}$$

where $U(a,b,x)$ is the confluent hypergeometric function of the second kind, whose small- and large-x behaviors are given in Eq. (76). The scaling function $g_2(x)$ is displayed in Fig. 8. The scaling function increases as $g_2(x) \sim \exp(-2/\sigma x)$ for small x and eventually decreases for large x as $g_2(x) \sim 1/x^{2\nu}$. In particular, for $\nu=1/2$ it has a very simple form $g_2(x)=\sqrt{2/\pi\sigma^3} x^{-3/2} \exp(-2/\sigma x)$.

C. Stable potential ($\mu<0$)

Following the analysis of Sec. IV C, in the limit $\alpha \to 0$, keeping $\alpha T=s$ fixed one gets

$$\overline{\exp[-\lambda(\alpha)T]} = \frac{4}{(2\sigma)^\nu \Gamma^2(\nu)} \left[s^{\nu/2} K_\nu\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right)\right]^2, \tag{96}$$

with $\nu=|\mu|/\sigma$.

On the other hand, in the corresponding limit $T\to\infty$, $t\to\infty$, but keeping $t/T=x$ fixed, using the scaling form

$$\overline{I_{\rm loc}(t|T)} = \frac{1}{T} g_3(t/T), \tag{97}$$

one gets

$$\int_0^\infty dt e^{-\alpha t}\overline{I_{\rm loc}(t|T)} = \int_0^\infty dx e^{-sx} g_3(x), \tag{98}$$

with $s=\alpha T$. Therefore, in this scaling limit Eq. (83) becomes

$$\int_0^\infty dx e^{-sx} g_3(x) = \frac{4}{(2\sigma)^\nu \Gamma^2(\nu)} \left[s^{\nu/2} K_\nu\left(\frac{\sqrt{2s}}{\sqrt{\sigma}}\right)\right]^2. \tag{99}$$

Now comparing the above equation with Eq. (94) one gets

$$g_3(x) = \sigma^{2\nu} g_2(x), \tag{100}$$

where the scaling function $g_2(x)$ is given by Eq. (95) and displayed in Fig. 8. While $\overline{I_{\rm loc}(t|T)}$ has the same scaling function (up to a multiplicative factor of $\sigma^{2\nu}$) for both stable and unstable potentials, the physical behaviors, however, are quite different. For the stable potential, $\overline{I_{\rm loc}(t|T)}$ is normalized to unity. Note that the scaling function $g_3(x)$ becomes narrower as one increases ν, as expected since the particle becomes more localized near the origin. For the unstable potential, on the other hand, the weight of $\overline{I_{\rm loc}(t|T)}$ decreases as $(\sigma T)^{-2\nu}$, as one increases ν, as expected since when the repulsive force from the origin becomes stronger, the particle escapes to $\pm\infty$ with a higher probability.

VII. OCCUPATION TIME WITHOUT DISORDER ($\sigma=0$)

In this case $V(x)=\theta(x)$ corresponds to T in Eq. (2), being the occupation time in the region $x>0$, and $P(T|t,0)$ in Eq. (15), being $P_{\rm occ}(T|t)$—the PDF of the occupation time for a given observation time window of size t and the initial position of the particle $x(0)=0$. Again, as before, we need to solve the differential equation (16) for $x>0$ and $x<0$ separately and then match the solutions at $x=0$. The matching condition for the slope of the solutions is obtained by integrating Eq. (16) across $x=0$. Thus the matching conditions are

$$u_+(0) = u_-(0) = u(0), \quad u'_+(0) = u'_-(0), \tag{101}$$

where $u_\pm(x)$ satisfy the following differential equations:

$$\frac{1}{2} u''_+(x) + F(x) u'_+(x) - (\alpha+p) u_+(x) = -1, \tag{102}$$

for $x>0$, and

$$\frac{1}{2} u''_-(x) + F(x) u'_-(x) - \alpha u_-(x) = -1, \tag{103}$$

for $x<0$. The boundary conditions of $u_\pm(x)$ when $x\to\pm\infty$ are obtained from the fact that if the starting position goes to $\pm\infty$, the particle will never cross the origin in finite time, $P(T|t,x\to\infty)=\delta(t-T)$ and $P(T|t,x\to-\infty)=\delta(T)$, and hence, from Eq. (15),

$$u_+(\infty) = \frac{1}{\alpha+p}, \quad u_-(-\infty) = \frac{1}{\alpha}. \tag{104}$$

Writing $u_+(x)=1/(\alpha+p)+B_+y_+(x)$ and $u_-(x)=1/\alpha+B_-y_-(x)$, we obtain the homogeneous differential equations for $y_\pm(x)$ as

$$\frac{1}{2}y_+''(x)+F(x)y_+'(x)-(\alpha+p)y_+(x)=0, \qquad (105)$$

for $x>0$, and

$$\frac{1}{2}y_-''(x)+F(x)y_-'(x)-\alpha y_-(x)=0, \qquad (106)$$

for $x<0$, with the boundary conditions $y_+(x\to\infty)=0$ and $y_-(x\to-\infty)=0$. The constants $B_\pm$ are determined by the matching conditions given in Eq. (101), which can be rewritten as

$$\frac{1}{\alpha+p}+B_+y_+(0)=\frac{1}{\alpha}+B_-y_-(0)=u(0), \qquad (107a)$$

$$B_+y_+'(0)=B_-y_-'(0). \qquad (107b)$$

Eliminating the constants from Eq. (107), we obtain the double Laplace transform of the PDF of the occupation time,

$$u(0)=\int_0^\infty dte^{-\alpha t}\int_0^t dTe^{-pT}P_{\text{occ}}(T|t)=\frac{\ell_1(\alpha,p)}{\alpha}+\frac{\ell_2(\alpha,p)}{\alpha+p}, \qquad (108)$$

where

$$\ell_1(\alpha,p)=\left[\frac{z_-(0)}{z_-(0)-z_+(0)}\right], \qquad (109)$$

$$\ell_2(\alpha,p)=\left[\frac{-z_+(0)}{z_-(0)-z_+(0)}\right], \qquad (110)$$

and $z_\pm(x)=y_\pm'(x)/y_\pm(x)$. Note that

$$\ell_1(\alpha,p)+\ell_2(\alpha,p)=1. \qquad (111)$$

Putting $p=0$ in Eq. (108) gives $u(0)=1/\alpha$, and hence inverting the Laplace transform with respect to α readily verifies the normalization

$$\int_0^t P_{\text{occ}}(T|t)dT=1. \qquad (112)$$

For any symmetric deterministic potential the distribution of the occupation time is symmetric about its mean $\langle T\rangle=t/2$—i.e., $P_{\text{occ}}(T|t)=P_{\text{occ}}(t-T|t)$. Then, it follows from this symmetry that

$$\ell_1(\alpha+p,-p)=\ell_2(\alpha,p). \qquad (113)$$

In other words, the double integral in Eq. (108) remains invariant under the following simultaneous replacements: $(\alpha+p)\to\alpha$ and $\alpha\to(\alpha+p)$. Thus under these replacements one must have $z_+(0)\to-z_-(0)$ and vice versa, which also implies that $z_+(0)=-z_-(0)$ for $p=0$. Equivalently, $\ell_1(\alpha,0)=\ell_2(\alpha,0)=1/2$, which also directly follows from Eqs. (111) and (113).

Therefore if one splits the distribution function into two parts, $P_{\text{occ}}(T|t)=R_L(T|t)+R_R(T|t)$, such that

$$\int_0^\infty dte^{-\alpha t}\int_0^t dTe^{-pT}R_L(T|t)=\frac{\ell_1(\alpha,p)}{\alpha}, \qquad (114)$$

$$\int_0^\infty dte^{-\alpha t}\int_0^t dTe^{-pT}R_R(T|t)=\frac{\ell_2(\alpha,p)}{\alpha+p}, \qquad (115)$$

then it follows from the above discussion that $R_L(t-T|t)=R_R(T|t)$. This symmetry of the distribution will come in handy later. Moreover, putting $p=0$ and inverting the Laplace transforms with respect to α gives the normalization for each part separately:

$$\int_0^t R_L(T|t)dT=\int_0^t R_R(T|t)dT=\frac{1}{2}. \qquad (116)$$

As an example, we first consider the pure case $\sigma=0$ in the force given by Eq. (6). For $F(x)=\mu\,\text{sgn}(x)$, the solutions of Eqs. (105) and (106) are obtained as

$$y_+(x)=y_+(0)\exp\{-[\mu+\sqrt{\mu^2+2(\alpha+p)}]x\}, \qquad (117)$$

for $x>0$, and

$$y_-(x)=y_-(0)\exp[(\mu+\sqrt{\mu^2+2\alpha})x], \qquad (118)$$

for $x<0$. These give the expressions for $z_\pm(0)=y'_\pm(0)/y_\pm(0)$ as

$$z_+(0)=-[\mu+\sqrt{\mu^2+2(\alpha+p)}],$$

$$z_-(0)=[\mu+\sqrt{\mu^2+2\alpha}]. \qquad (119)$$

In the following subsections we will consider the three different cases (i) $\mu=0$, (ii) $\mu>0$, and (iii) $\mu<0$.

A. Flat potential ($\mu=0$)

For $\mu=0$, using $z_+(0)=-\sqrt{2(\alpha+p)}$ and $z_-(0)=\sqrt{2\alpha}$ from Eq. (108) we get

$$\int_0^\infty dte^{-\alpha t}\int_0^t dTe^{-pT}P_{\text{occ}}(T|t)=\frac{1}{\sqrt{\alpha(\alpha+p)}}. \qquad (120)$$

Inverting the double Laplace transform with respect to p and then with respect to α finally reproduces the well-known Lévy's "arcsine" law [11] for the PDF of the occupation time of an ordinary Brownian motion,

$$P_{\text{occ}}(T|t)=\frac{1}{\pi\sqrt{T(t-T)}}, \quad 0<T<t. \qquad (121)$$

The distribution $P_{\text{occ}}(T|t)$ diverges on both ends $T=0$ and $T=t$, which indicates that the Brownian particle "tends" to stay on one side of the origin.

B. Unstable potential ($\mu>0$)

Since for $\mu>0$ the force is repulsive from the origin $x=0$, one would expect the occupation time distribution to be convex (concave upward), with minimum at $T=t/2$. Now in the limit of large window size, $t\to\infty$, the part of the distribution $P_{\mathrm{occ}}(T|t)$ to the left of the midpoint $T=t/2$ approaches $R_L(T|t)$, as the midpoint itself goes to ∞.

By making a change of variable $z=\alpha t$, it follows, from Eq. (114)

$$\int_0^\infty dz e^{-z}\int_0^{z/\alpha} dT e^{-pT} R_L(T|z/\alpha) = \ell_1(\alpha,p). \tag{122}$$

Now the large-t limit of $R_L(T|t)$ can be obtained by taking $\alpha\to 0$ in the above equation, where one realizes that $R_L(T|t)$ approaches a steady (t-independent) distribution, $R_L(T|t\to\infty)\to R_L(T)$, whose Laplace transform is given by

$$\int_0^\infty dT e^{-pT} R_L(T) = \ell_1(0,p), \tag{123}$$

where $\ell_1(0,p)$ is obtained from Eq. (109), by using $z_\pm(0)$ from Eq. (119), which gives

$$\ell_1(0,p) = \frac{2\mu}{3\mu + \sqrt{\mu^2+2p}}. \tag{124}$$

The above Laplace transform can be inverted with respect to p, which gives

$$R_L(T) = \mu\sqrt{2} e^{-\mu^2 T/2} \times \left[\frac{1}{\sqrt{\pi T}} - \frac{3\mu}{\sqrt{2}} \exp\left(\frac{9\mu^2}{2}T\right)\mathrm{erfc}\left(\frac{3\mu}{\sqrt{2}}\sqrt{T}\right)\right], \tag{125}$$

with the normalization $\int_0^\infty R_L(T)dT = \ell_1(0,0)=1/2$.

The limiting behavior of the distribution is given by

$$R_L(T) \approx \frac{\mu\sqrt{2}}{\sqrt{\pi T}}, \tag{126}$$

for small T, and decays exponentially for large T,

$$R_L(T) \approx \frac{\sqrt{2}}{9\mu\sqrt{\pi}} \frac{e^{-\mu^2 T/2}}{T^{3/2}}. \tag{127}$$

C. Stable potential ($\mu<0$)

As we discussed earlier in Sec. III C in the context of the local time, for a generic stable potential $U(x)$ the system eventually becomes ergodic at large t and hence the average $\langle\theta[x(t)]\rangle$ approaches its stationary value $\langle\theta[x(t)]\rangle\to Z_+/Z$, where $Z=\int_{-\infty}^{\infty} e^{-2U(x)}dx$ is the equilibrium partition function and $Z_+=\int_0^\infty e^{-2U(x)}dx$ is the restricted partition function. Therefore, for large t the average occupation time $\langle T\rangle=\int_0^t\langle\theta[x(t')]\rangle dt'$ scales linearly with t:

$$\langle T\rangle \to \left(\frac{Z_+}{Z}\right)t. \tag{128}$$

Note that when the potential $U(x)$ is symmetric about zero, the average occupation time $\langle T\rangle = t/2$ for all t.

From the definition $T-\langle T\rangle=\int_0^t\{\theta[x(t')]-\langle\theta[x(t')]\rangle\}dt'$, it follows that when $T\to\langle T\rangle$, the random variables $\theta[x(t')]-\langle\theta[x(t')]\rangle$ at different times t' become only weakly correlated. Then, in the limit when t is much larger than the correlation time between these variables, one expects the central limit theorem to hold, which predicts a Gaussian form for the distribution of the occupation time T near the mean value $\langle T\rangle$,

$$P_{\mathrm{occ}}(T|t) \sim \exp\left[-\frac{(T-\langle T\rangle)^2}{2\Sigma^2}\right], \tag{129}$$

where the variance $\Sigma^2=\langle T^2\rangle-\langle T\rangle^2$ can be obtained from the Laplace transform of the moments,

$$\int_0^\infty \langle T^n\rangle e^{-\alpha t}dt = (-1)^n \left.\frac{\partial^n u(0)}{\partial p^n}\right|_{p=0}, \tag{130}$$

with $u(0)$ given by Eq. (108).

For the particular attractive force $F(x)=-|\mu|\,\mathrm{sgn}(x)$, using $z_\pm(0)$ from Eq. (119) in Eq. (108) and taking derivatives with respect to p we get

$$-\left.\frac{\partial u(0)}{\partial p}\right|_{p=0} = \frac{1}{2\alpha^2}, \tag{131}$$

$$\left.\frac{\partial^2 u(0)}{\partial p^2}\right|_{p=0} = \frac{1}{2\alpha^3} + \frac{1}{4\mu^2\alpha^2} + O\left(\frac{1}{\alpha}\right). \tag{132}$$

Therefore inverting the Laplace transform in Eq. (130) with respect to α immediately gives $\langle T\rangle=t/2$ for all t and $\langle T^2\rangle=t^2/4+t/4\mu^2$ for large t which gives $\Sigma^2=t/4\mu^2$.

VIII. OCCUPATION TIME WITH DISORDER ($\sigma>0$)

Now we consider the occupation time when the disorder is switched on: $\sigma>0$ in Eq. (6). Our aim is to calculate the disorder-averaged $\overline{P_{\mathrm{occ}}(T|t)}$. As one realizes from Eqs. (108)–(110), to calculate $\overline{P_{\mathrm{occ}}(T|t)}$ one needs the distribution of $-z_+(0)$ and $z_-(0)$, which are given by Eqs. (B9) and (B11) with $a_+=\alpha+p$ and $a_-=\alpha$, respectively. In the following subsections, we will consider the three cases (i) $\mu=0$, (ii) $\mu>0$, and (iii) $\mu<0$.

A. Flat potential ($\mu=0$): Sinai model

We first consider the diffusive motion of a particle in a continuous Sinai potential, where the potential itself is a Brownian motion in space. In the limit of large window size t the left half of the disorder-averaged PDF of the occupation time $\overline{R_L(T|t)}$ for $0\le T\le t/2$ is obtained by taking the disorder average in Eq. (114). The right half of the distribution for $t/2\le T\le t$ is just the symmetric reflection of the left part. The detailed calculations for $\overline{R_L(T|t)}$ are presented in Appendix C.

We find that $\overline{R_L(T|t)}$ has a large-t behavior,

$$\overline{R_L(T|t)} \approx \frac{1}{\ln t}R(T), \tag{133}$$

where the function $R(T)$ is independent of t. The limiting behaviors of $R(T)$ are given by

$$R(T) \approx \frac{\sqrt{2}\sigma}{\sqrt{\pi T}}, \tag{134}$$

as $T \to 0$, and

$$R(T) \sim \frac{1}{2T}, \tag{135}$$

for large T.

B. Unstable potential ($\mu > 0$)

For $\mu > 0$, we find that disorder does not change the asymptotic behavior of the distribution for the pure case qualitatively. The calculations are presented in Appendix D. We find that in the limit $t \to \infty$ the left half of the disorder-averaged occupation time distribution tends to a t-independent form

$$\overline{R_L(T|t)} = \overline{R_L(T)}. \tag{136}$$

In fact, the small-T limit of $\overline{R_L(T)}$ remains the same as in the pure case:

$$\overline{R_L(T)} \approx \frac{\mu\sqrt{2}}{\sqrt{\pi T}}. \tag{137}$$

For large T, the distribution $\overline{R_L(T)}$ still decays exponentially,

$$\overline{R_L(T)} \sim e^{-bT}, \tag{138}$$

where the decay coefficient b is, however, different from the pure case [see Eq. (D17)].

C. Stable potential ($\mu < 0$)

This particular situation, where one finds the interplay between two competing processes, is a very interesting one. On the one hand, as we discussed in Sec. VII C, the stable potential in the absence of the disordered potential makes the system ergodic in the large-t limit, and as a result the PDF of the occupation time is peaked at $=t/2$ and decays fast away from it. On the other hand, as we discussed in Sec. VIII A, without any underlying deterministic potential the disorder-averaged PDF of the occupation time is convex (concave upward) with a minimum at $T=t/2$ and diverges at both ends $T \to 0$ and $T \to t$. Therefore, if both the stable potential and disordered potential are included, as their relative strength $\nu = |\mu|/\sigma$ is varied, one expects a phase transition at some critical value ν_c where the system loses ergodicity.

In the scaling limit where both $t \to \infty$ and $T \to \infty$, but their ratio $y = T/t$ is kept fixed, we find that the disorder-averaged PDF of the occupation time has a scaling form

$$\overline{P_{\rm occ}(T|t)} = \frac{1}{t} f_0(T/t). \tag{139}$$

The calculation of the scaling function $f_0(y)$ is presented in Appendix E, where we find the beta law

$$f_0(y) = \frac{1}{B(\nu,\nu)}[y(1-y)]^{\nu-1}, \quad 0 \leq y \leq 1, \tag{140}$$

where $\nu = |\mu|/\sigma$ and $B(\nu,\nu)$ is the beta function [44]. Now, if one tunes the parameter ν by varying either μ or the disorder

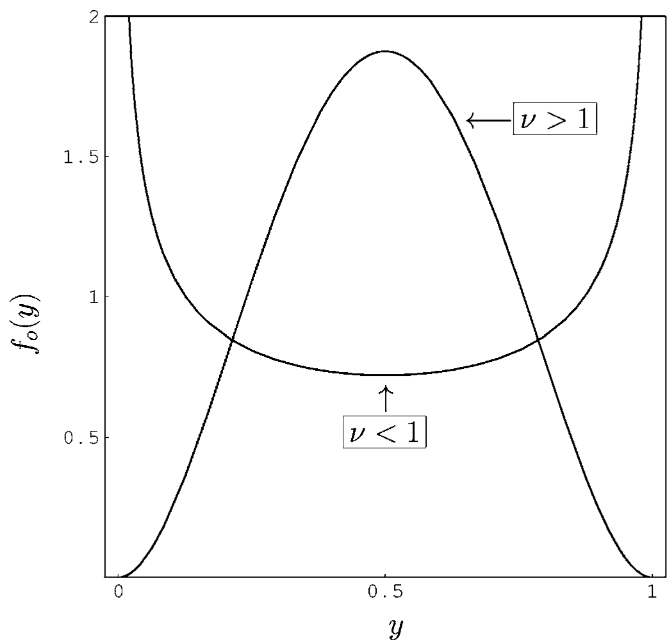

FIG. 9. The scaling functions $f_0(x)$ in Eq. (139) plotted by using Eq. (140).

strength σ, the distribution $\overline{P_{\rm occ}(T|t)}$ exhibits a phase transition in the ergodicity of the particle position at $\nu_c = 1$ (Fig. 9). For $\nu < \nu_c$ the distribution $f_0(y)$ in Eq. (139) is convex with a minimum at $y=1/2$ and diverges at the two ends $y=0,1$. This means that the particle tends to stay on one side of the origin such that T is close to either 0 or t. In other words the paths with a small number of zero crossings carry more weight than the ones that cross many times. For $\nu > \nu_c$ the scenario is exactly opposite, where $f_0(y)$ is maximum at the mean value $y=1/2$, indicating that the particle tends to spend equal times on both sides of the origin $x=0$, such that paths with a large number of zero crossings, for which T is closer to $t/2$, carry larger weight. A similar phase transition in the ergodicity properties of a stochastic process as one changes a parameter was first noted in the context of a diffusion equation [19] and later found for a class of Gaussian Markov processes [20] and in simple models of coarsening [47,48].

A very interesting observation about Eq. (140) is that for $\nu = 1/2$, the result is the same as Lévy's result for the one-dimensional Brownian motion given by Eq. (121). It seems as if the attractive force cancels the effect of disorder exactly at $\nu = 1/2$. However, this is no longer true in the context of the local time.

IX. INVERSE OCCUPATION TIME WITHOUT DISORDER ($\sigma = 0$)

In this case $I(t|T,0)$ in Eq. (21) is replaced with $I_{\rm occ}(t|T)$, which is the distribution of the time t needed to observe the particle with a starting position $x=0$, until the total amount of time spent on the positive side $x>0$ is T. The corresponding $u(0)$ in Eq. (21) with $x=0$, which is the double Laplace transform of the PDF of the occupation time, has already been evaluated in Sec. VII, which is given by Eq. (108). Substituting $u(0)$ in Eq. (21) gives

$$\int_0^\infty dTe^{-pT}\int_T^\infty dte^{-\alpha t}I_{\rm occ}(t|T)=\frac{\ell_2(\alpha,p)}{\alpha+p}, \qquad (141)$$

where $\ell_2(\alpha,p)$ is given by Eq. (110). Comparing the above equation with Eq. (115), one can infer that $I_{\rm occ}(t|T)$ and $R_R(T|t)$ have the same functional form—i.e., $I_{\rm occ}(t|T)=R_R(T|t)$ and especially for the symmetric deterministic potential $I_{\rm occ}(t|T)=R_R(T|t)=R_L(t-T|t)$.

It is useful to present the above equation in the following form:

$$\int_0^\infty dze^{-z}\int_0^\infty d\tau e^{-\alpha\tau}I_{\rm occ}\left(\tau+\frac{z}{\beta}\Big|\frac{z}{\beta}\right)=\ell_1(\beta,\alpha-\beta), \qquad (142)$$

which has been obtained by substituting $p=\beta-\alpha$ in Eq. (141) and subsequently making the change of variables $\beta T=z$ and $\tau=t-T$. On the right-hand side, we have substituted $\ell_2(\alpha,\beta-\alpha)=\ell_1(\beta,\alpha-\beta)$, using Eq. (113), and $\ell_1(\alpha,p)$ is given by Eq. (109). Now by taking the limit $\beta\to 0$ in Eq. (142), one obtains the large-T behavior of $I_{\rm occ}(t|T)$.

For the pure case, $\sigma=0$ in Eq. (6), we have already obtained $z_\pm(0)$ in Sec. VII which are given by Eq. (119) and hence we can evaluate $\ell_1(\alpha,p)$ and $\ell_2(\alpha,p)$ by using Eqs. (109) and (110), respectively. In the following subsections we will analyze the behavior of $I_{\rm occ}(t|T)$ for the cases (i) $\mu=0$, (ii) $\mu>0$, and (iii) $\mu<0$.

A. Flat potential ($\mu=0$)

For $\mu=0$, which is the case of a simple Brownian motion, $z_+(0)=-\sqrt{2(\alpha+p)}$ and $z_-(0)=\sqrt{2\alpha}$. Therefore, using Eq. (110), from Eq. (141) we get

$$\int_0^\infty dTe^{-pT}\int_T^\infty dte^{-\alpha t}I_{\rm occ}(t|T)=\frac{1}{\sqrt{\alpha+p}(\sqrt{\alpha}+\sqrt{\alpha+p})}. \qquad (143)$$

Now inverting the Laplace transform with respect to p gives

$$\int_T^\infty dte^{-\alpha t}I_{\rm occ}(t|T)={\rm erfc}(\sqrt{\alpha T}), \qquad (144)$$

and further inverting the Laplace transform with respect to α gives

$$I_{\rm occ}(t|T)=\frac{\sqrt{T}}{\pi t\sqrt{t-T}}\theta(t-T), \qquad (145)$$

with the normalization condition $\int_T^\infty I_{\rm occ}(t|T)dt=1$, which is readily checked by putting $\alpha=0$ in Eq. (144). The inverse occupation time has nonzero support only for $t>T$, as shown in Fig. 10.

Note that, since $R_R(T|t)=I_{\rm occ}(t|T)=\sqrt{T}/\pi t\sqrt{t-T}$ and $R_L(T|t)=R_R(t-T|t)=\sqrt{t-T}/\pi t\sqrt{T}$, adding the two parts, $P_{\rm occ}(T|t)=R_L(T|t)+R_R(T|t)=1/\pi\sqrt{T(t-T)}$, one recovers Eq. (121).

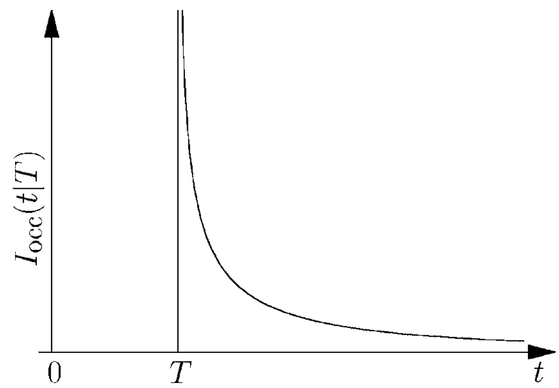

FIG. 10. The PDF of the inverse occupation time for simple Brownian motion.

B. Unstable potential ($\mu>0$)

For $\mu>0$, although one can invert the Laplace transform in Eq. (141) with respect to p exactly; the other Laplace transform with respect to α can be inverted only in the large-T limit. Therefore, to keep the presentation simpler, we will consider the large-T behavior of $I_{\rm occ}(t|T)$ by analyzing Eq. (142) in the limit $\beta\to 0$.

By using $z_\pm(0)$ from Eq. (119) in Eq. (109), one gets

$$\ell_1(0,\alpha)=\frac{2\mu}{3\mu+\sqrt{\mu^2+2\alpha}}. \qquad (146)$$

Therefore, Eq. (142) suggests that $I_{\rm occ}(t|T)$ should only depend on the difference $(t-T)$ at large T:

$$I_{\rm occ}(t|T)=I_1(t-T). \qquad (147)$$

Substituting this form in Eq. (142) in the limit $\beta\to 0$ gives

$$\int_0^\infty d\tau e^{-\alpha\tau}I_1(\tau)=\ell_1(0,\alpha), \qquad (148)$$

where putting $\alpha=0$ gives the normalization $\int_0^\infty I_1(\tau)d\tau=\ell_1(0,0)=1/2$, indicating that the particle can escape to $-\infty$ with probability $1/2$ for the unstable potential (the force is repulsive from the origin). Now inverting the Laplace transform with respect to α gives

$$I_1(\tau)=\mu\sqrt{2}e^{-\mu^2\tau/2}\left[\frac{1}{\sqrt{\pi\tau}}-\frac{3\mu}{\sqrt{2}}\exp\left(\frac{9\mu^2}{2}\tau\right){\rm erfc}\left(\frac{3\mu}{\sqrt{2}}\sqrt{\tau}\right)\right], \qquad (149)$$

The limiting behavior of this distribution is given by

$$I_1(\tau)\approx\frac{\mu\sqrt{2}}{\sqrt{\pi\tau}}, \qquad (150)$$

for small $\tau=(t-T)$, and decays exponentially for large $\tau=(t-T)$,

$$I_1(\tau)\approx\frac{\sqrt{2}}{9\mu\sqrt{\pi}}\frac{e^{-\mu^2\tau/2}}{\tau^{3/2}}. \qquad (151)$$

C. Stable potential ($\mu<0$)

It is reasonable to consider the difference variable $t-T$ instead of t, as $t\geqslant T$. Therefore, we write

$$I_{\rm occ}(t|T)=I_2(t-T,T). \tag{152}$$

Substituting this form and $p=\beta-\alpha$ in Eq. (141) one gets

$$\int_0^\infty dTe^{-\beta T}\int_0^\infty d\tau e^{-\alpha\tau}I_2(\tau,T)=\frac{\ell_1(\beta,\alpha-\beta)}{\beta}, \tag{153}$$

where we have substituted $\ell_2(\alpha,\beta-\alpha)=\ell_1(\beta,\alpha-\beta)$ on the right-hand side, using Eq. (113). Using $z_\pm(0)$ from Eq. (119) for $\mu<0$ in Eq. (109) gives

$$\ell_1(\beta,\alpha-\beta)=\left[\frac{\sqrt{\mu^2+2\beta}-|\mu|}{\sqrt{\mu^2+2\beta}+\sqrt{\mu^2+2\alpha}-2|\mu|}\right], \tag{154}$$

Therefore, taking the small-β limit in Eq. (153) gives

$$\int_0^\infty dTe^{-\beta T}\int_0^\infty d\tau e^{-\alpha\tau}I_2(\tau,T)=\frac{1}{\beta+|\mu|\sqrt{\mu^2+2\alpha}-\mu^2}, \tag{155}$$

and inverting the Laplace transform with respect to β gives

$$\int_0^\infty d\tau e^{-\alpha\tau}I_2(\tau,T)=\exp(\mu^2T-|\mu|T\sqrt{\mu^2+2\alpha}), \tag{156}$$

where putting $\alpha=0$ confirms the normalization condition $\int_0^\infty I_2(\tau,T)d\tau=1$. Now by inverting the other Laplace transform with respect to α one gets the distribution

$$I_2(\tau,T)=\frac{|\mu|T}{\sqrt{2\pi\tau^3}}e^{-\mu^2(\tau-T)^2/2\tau}, \tag{157}$$

where $\tau=t-T$.

X. INVERSE OCCUPATION TIME WITH DISORDER ($\sigma>0$)

In the presence of disorder—i.e., $\sigma>0$ in Eq. (6)—taking the disorder average in Eq. (141) gives

$$\int_0^\infty dTe^{-pT}\int_T^\infty dte^{-\alpha t}\overline{I_{\rm occ}(t|T)}=\frac{\overline{\ell_2(\alpha,p)}}{\alpha+p}, \tag{158}$$

where $\overline{\ell_2(\alpha,p)}$ is obtained by taking the disorder average of Eq. (110), using the distributions of $-z_+(0)$ and $z_-(0)$ given by Eqs. (B9) and (B11), respectively, with $a_+=\alpha+p$ and $a_-=\alpha$.

It is useful to consider a different form of the above equation, which is obtained by taking the disorder average of Eq. (142),

$$\int_0^\infty dze^{-z}\int_0^\infty d\tau e^{-\alpha\tau}\overline{I_{\rm occ}\left(\tau+\frac{z}{\beta}\Big|\frac{z}{\beta}\right)}=\overline{\ell_1(\beta,\alpha-\beta)}, \tag{159}$$

where by taking the limit $\beta\to0$, one obtains the large-T behavior of $\overline{I_{\rm occ}(t|T)}$.

A. Flat potential ($\mu=0$): Sinai model

We will now study the large-T behavior of $\overline{I_{\rm occ}(t|T)}$, for the Sinai potential ($\mu=0$), by analyzing Eq. (159) in the limit $\beta\to0$.

It follows from Eq. (C2) that

$$\overline{\ell_1(\beta,\alpha-\beta)}=\frac{m_1(\beta,\alpha-\beta)}{\Omega_+\Omega_-}, \tag{160}$$

where $m_1(\alpha,p)$ is given by Eq. (C3) and

$$\Omega_+=2K_0\left(\frac{\sqrt{2\beta}}{\sigma}\right),\quad \Omega_-=2K_0\left(\frac{\sqrt{2\alpha}}{\sigma}\right). \tag{161}$$

In the limit $\beta\to0$, since $\Omega_+\sim-\ln\beta$; hence,

$$\overline{\ell_1(\beta,\alpha-\beta)}\sim\frac{m_1(0,\alpha)}{[-2K_0(\sqrt{2\alpha}/\sigma)\ln\beta]}, \tag{162}$$

which suggests the following scaling form at large T:

$$\overline{I_{\rm occ}(t|T)}=\frac{1}{\ln T}I_3(t-T). \tag{163}$$

Therefore, in the limit $\beta\to0$, substituting the above scaling form in Eq. (159) and using Eq. (162) one gets

$$\int_0^\infty d\tau e^{-\alpha\tau}I_3(\tau)=\frac{m_1(0,\alpha)}{2K_0(\sqrt{2\alpha}/\sigma)}. \tag{164}$$

However, the above Laplace transform is the same one given by Eq. (C11) in Appendix C, where α is replaced by p. Therefore we can directly borrow the results obtained there. Using the results from Eq. (C19) gives

$$I_3(\tau)\approx\frac{\sqrt{2}\sigma}{\sqrt{\pi\tau}}\quad\text{as }\tau\to0, \tag{165}$$

and results from Eq. (C26) give

$$I_3(\tau)\sim\frac{1}{2\tau}\quad\text{as }\tau\to\infty. \tag{166}$$

B. Unstable potential ($\mu>0$)

For $\mu>0$, Eq. (159) suggests that in the large-T limit, $\overline{I_{\rm occ}(t|T)}$ will only depend on the difference $(t-T)$,

$$\overline{I_{\rm occ}(t|T)}=I_4(t-T). \tag{167}$$

Therefore, in the limit $\beta\to0$, using the above form in Eq. (159) one gets

$$\int_0^\infty d\tau e^{-\alpha\tau}I_4(\tau)=\overline{\ell_1(0,\alpha)}. \tag{168}$$

However, the above Laplace transform is the same one given by Eq. (D3) where α is replaced by p. Therefore borrowing the results from Appendix D readily gives

$$I_4(\tau)\approx\frac{\mu\sqrt{2}}{\sqrt{\pi\tau}}, \tag{169}$$

for small τ, and

$$I_4(\tau) \sim e^{-b\tau}, \tag{170}$$

for large τ, with the same constant b as in Eq. (D16).

The normalization condition $\int_0^\infty I_4(\tau)d\tau = \ell_1(0,0) = 1/2$ indicates that the particle escapes to $-\infty$ with probability 1/2.

C. Stable potential ($\mu < 0$)

We are interested in the behavior of $\overline{I_{\text{occ}}(t|T)}$ in the scaling limit where $t \to \infty$ and $T \to \infty$, but the ratio $x = t/T$ is kept fixed. Substituting $T = z/\alpha$, $t = xz/\alpha$, and $p = \alpha s$ in Eq. (158), we get

$$\int_1^\infty dx \int_0^\infty dz e^{-(s+x)z} \left[\frac{z}{\alpha} \overline{I_{\text{occ}}\left(\frac{xz}{\alpha} \Big| \frac{z}{\alpha} \right)} \right] = \frac{1 - m_3(\alpha, s)}{1 + s}, \tag{171}$$

where $m_3(\alpha,s) = \overline{\ell_1(\alpha,\alpha s)} = 1 - \overline{\ell_2(\alpha,p)}$, given by Eq. (E6). The above equation suggests the scaling form

$$\overline{I_{\text{occ}}(t|T)} = \frac{1}{T} g_0(t/T), \tag{172}$$

with the normalization $\int_1^\infty g_0(x)dx = 1$, which follows directly from the normalization $\int_T^\infty \overline{I_{\text{occ}}(t|T)}dt = 1$. By substituting the above scaling form in Eq. (171) in the limit $\alpha \to 0$, after simplification one gets

$$\int_1^\infty \left[\frac{x-1}{x+s} \right] g_0(x) dx = m_3(0,s), \tag{173}$$

where $m_3(0,s)$ is given by Eq. (E16). By making a change of variable $y = 1/x$, Eq. (E16) reads

$$m_3(0,s) = \frac{1}{B(\nu,\nu)} \int_1^\infty \left[\frac{x-1}{x+s} \right] \frac{(x-1)^{\nu-1}}{x^{2\nu}} dx. \tag{174}$$

Therefore, comparing Eqs. (173) and (174) readily gives the inverted beta law

$$g_0(x) = \frac{1}{B(\nu,\nu)} \frac{(x-1)^{\nu-1}}{x^{2\nu}}, \quad x > 1, \tag{175}$$

which is displayed in Fig. 11. The scaling function $g_0(x)$ has a maximum at $x = 2\nu/(\nu+1)$ for $\nu > 1$. However, $g_0(x)$ diverges near $x = 1$ for $\nu < 1$. Note that for $\nu = 1/2$, Eq. (175) gives identical results to that of a pure Brownian motion ($\mu = 0$ and $\sigma = 0$), given by Eq. (145).

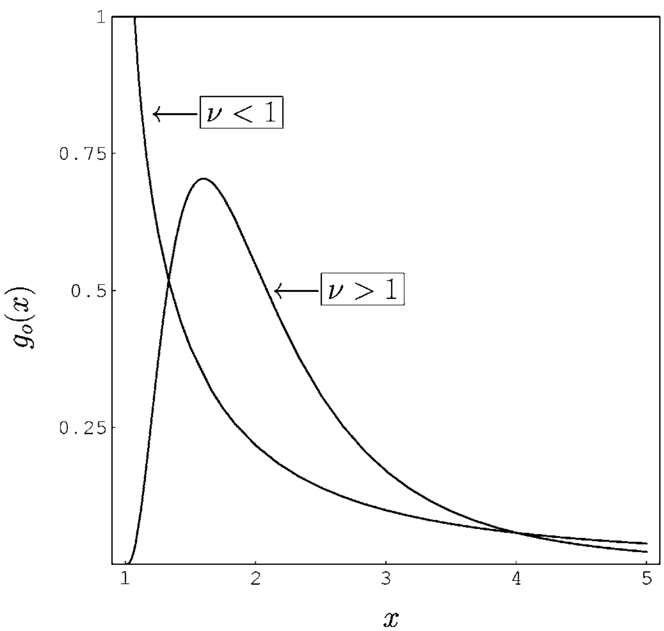

FIG. 11. The scaling functions $g_0(x)$ in Eq. (172) plotted by using Eq. (175).

XI. CONCLUDING REMARKS

In this paper we have considered the motion of a particle in a one-dimensional random potential. We have presented a general formalism for computing statistical properties of functionals and the inverse functionals of this process. We have used a backward Fokker-Planck equation approach to calculate the PDF of these functionals for each realization of the quenched random potential. The most difficult part of the problem is to carry out the disorder average on these PDF's. Thus to demonstrate the formalism explicitly, we have chosen the external potential to be the combination of a deterministic part and a random part, $U(x) = -\mu|x| + \sqrt{\sigma}B(x)$, where $B(x)$ is the trajectory of a Brownian motion in space. The case $\mu = 0$ in the potential corresponds to the Sinai model. The deterministic part of the external potential is stable for $\mu < 0$ and unstable for $\mu > 0$. The PDF's of the functional and the inverse functional vary from one realization of $B(x)$ to another, and in this paper we have shown how to carry out the disorder average on them, for two particular functionals: namely, the local time and the occupation time, and their inverse. Despite the simplicity of the model, we get very rich and interesting behaviors by tuning the parameter μ/σ, which we have summarized in Tables I–III, for $\mu = 0$, $\mu > 0$, and $\mu < 0$, respectively. In many cases the disorder changes the behavior of the PDF drastically from the pure case ($\sigma = 0$).

A very interesting phase transition in the ergodicity of the particle position occurs at a critical value of the parameter, $|\mu|/\sigma = 1$, when the deterministic part of the potential is stable ($\mu < 0$). For $|\mu|/\sigma < 1$, when the particle gets trapped in the wells of the random potential, the deterministic force $-|\mu|\text{sgn}(x)$ is not strong enough to lift it from the well and push it towards the origin and hence there are a small number of zero crossings. On the other hand, for $|\mu|/\sigma > 1$, the strong deterministic force sends the particle frequently towards the origin, and hence the system becomes ergodic. This change in the ergodic properties shows up in the qualitative change in the curvatures of the disorder-averaged PDF's when the parameter $\nu = |\mu|/\sigma$ passes through unity. While for $\nu < 1$ the disorder-averaged PDF of the occupation time $\overline{P_{\text{occ}}(T|t)}$ is concave upward with a minimum at $T = t/2$ and diverges at both ends $T = 0$ and $T = t$, for $\nu > 1$ it is concave downward, which goes to zero at the two ends $T = 0$ and $T = t$, and has a maximum at $T = t/2$ (see Fig. 9). In the context of inverse occupation time, while for $\nu < 1$, the disorder-averaged PDF $\overline{I_{\text{occ}}(t|T)}$ diverges near its lower end

$t=T$ and decreases monotonically as t increases; for $\nu>1$, it has a maximum at $t=[2\nu/(\nu+1)]T$ and goes to zero at both ends $t=T$ and $t\to\infty$ (see Fig. 11). Similarly, the disorder-averaged PDF of the local time $\overline{P_{\rm loc}(T|t)}$ diverges near the lower end $T=0$ and decreases monotonically as T increases for $\nu<1$. On the other hand, for $\nu>1$, it has a maximum and goes to zero at both ends $T=0$ and $T\to\infty$ (see Fig. 5).

For the stable potential, another very interesting observation is that at $|\mu|/\sigma=1/2$, in the limit $T\to\infty$ and $t\to\infty$ while keeping the ratio T/t fixed, the exact asymptotic disorder-averaged PDF's of the occupation time $\overline{P_{\rm occ}(T|t)}$ and inverse occupation time $\overline{I_{\rm occ}(t|T)}$ become exactly identical to the respective PDF's $P_{\rm occ}(T|t)$ and $I_{\rm occ}(t|T)$ for the simple Brownian motion ($\mu=0$ and $\sigma=0$). It looks as if at the particular value $|\mu|/\sigma=1/2$, the effect of disorder is exactly canceled by the deterministic stable potential. However, a similar conclusion is not true in the context of the local time and inverse local time. Therefore, a physical understanding of what exactly happens at this particular value of the parameter will be extremely useful.

There are several directions open for pursuing research further in this area. In this paper we have considered only the average of the PDF's over disorder. However, in many cases, as we have seen in this paper, the disorder broadens the distributions considerably. For example, for the unstable potential ($\mu>0$), even though for each realization of random potential the local time has a narrow exponential distribution, by taking the disorder average one gets a broad power-law distribution, which is the indication of large sample-to-sample fluctuations and lack of self-averaging. Therefore, in this situations knowledge about the disorder-averaged PDF (first moment) is not enough, and one requires to compute the other higher moments (over disorder). Thus extending our formalism to compute the full distribution (over disorder) of the PDF will be very useful.

The random part of the potential we have considered in this paper is very particular, where the barrier heights grow as $\sqrt{x}$. However, in realistic systems the random potential remains of order 1 throughout the sample. Therefore, it will be very interesting to extend this formalism to more realistic random potentials.

Recently several asymptotically exact long-time results for other quantities in the Sinai model were obtained by using a real-space renormalization-group method [39]. Using that method, reproducing the exact results obtained in this paper remains a challenging open problem. Another interesting direction is to study the properties of functionals of a more general non-Markovian stochastic process in random media and to extend our results to higher dimensions.

APPENDIX A: PDF OF THE LOCAL TIME IN THE CASE OF THE STABLE POTENTIAL, $\mu<0$ AND $\sigma=0$ IN Eq. (6)

In this appendix we will derive the PDF of the local time $P_{\rm loc}(T|t)$, for the stable potential ($\mu<0$) in the absence of disorder ($\sigma=0$). In this case by solving Eq. (25) with the boundary conditions $y_+(x\to\infty)\to 0$ and $y_-(x\to-\infty)\to 0$ we get

$$y_\pm(x) = y_\pm(0)\exp[\mp(-\nu+\sqrt{\nu^2+2\alpha})x], \tag{A1}$$

where $\nu=|\mu|$. Substituting these results in Eq. (28) we get $\lambda(\alpha)=-\nu+\sqrt{\nu^2+2\alpha}$. Therefore the Laplace transform $G(\alpha)$ in Eq. (30) becomes

$$G(\alpha) = \frac{-\nu+\sqrt{\nu^2+2\alpha}}{\alpha}\exp[-(-\nu+\sqrt{\nu^2+2\alpha})T]. \tag{A2}$$

Now making a shift $\alpha=\beta-\nu^2/2$ in Eq. (30) yields

$$\int_0^\infty dt e^{-\beta t}[e^{\nu^2 t/2}P_{\rm loc}(T|t)] = \sqrt{2}e^{\nu T}\frac{e^{-\sqrt{2\beta}T}}{\sqrt{\beta}+\nu/\sqrt{2}}, \tag{A3}$$

where the right-hand side is the Laplace transform of $e^{\nu^2 t/2}P_{\rm loc}(T|t)$. Inverting the Laplace transform with respect to β and after simplification gives the exact distribution of the local time for all T and t,

$$P_{\rm loc}(T|t) = \frac{\sqrt{2}}{\sqrt{\pi t}}e^{-(T-\nu t)^2/2t} - \nu e^{2\nu T}\,\mathrm{erfc}\left(\frac{\nu}{2}\sqrt{t}+\frac{T}{\sqrt{2t}}\right), \tag{A4}$$

where erfc(x) is the complementary error function. Note that Eq. (A4) reduces to Eq. (33) for $\nu=0$.

For large t, since

$$\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\sqrt{t}+\frac{T}{\sqrt{2t}}\right) \sim \frac{1}{\sqrt{\pi}}\left[\frac{\nu}{\sqrt{2}}\sqrt{t}+\frac{T}{\sqrt{2t}}\right]^{-1} \times\exp\left(-\left[\frac{\nu}{\sqrt{2}}\sqrt{t}+\frac{T}{\sqrt{2t}}\right]^2\right), \tag{A5}$$

Eq. (A4) simplifies to

$$P_{\rm loc}(T|t) = \left[\frac{T}{T+\nu t}\right]\frac{\sqrt{2}}{\sqrt{\pi t}}e^{-(T-\nu t)^2/2t}. \tag{A6}$$

Putting $\nu=0$ in the above equation one still recovers the result given by Eq. (33). For nonzero ν, near the mean $\langle T\rangle=\nu t$, the PDF of the local time reduces to a Gaussian one

$$P_{\rm loc}(T|t) \approx \frac{1}{\sqrt{2\pi t}}e^{-(T-\nu t)^2/2t}. \tag{A7}$$

APPENDIX B: PDF OF THE SLOPE VARIABLES $Z_\pm(0)$ THAT APPEAR IN THE DISORDER-AVERAGE COMPUTATIONS

Both in the contexts of local and occupation time we have a homogeneous differential equation of the type

$$\frac{1}{2}y''_\pm(x) + F(x)y'_\pm(x) - a_\pm y_\pm(x) = 0, \tag{B1}$$

with the boundary conditions $y_+(x\to\infty)\to 0$ and $y_-(x\to-\infty)\to 0$, and the force

$$F(x) = \mu\,\mathrm{sgn}(x) + \sqrt{\sigma}\xi(x), \tag{B2}$$

with $\langle\xi(x)\rangle=0$ and $\langle\xi(x)\xi(x')\rangle=\delta(x-x')$. For each realization of $\{\xi(x)\}$ in the force $F(x)$, the solution of Eq. (B1) is different, and for the disorder-averaged computations performed in this paper we finally require the distributions of the stochastic variables $y'_{\pm}(0)/y_{\pm}(0)$. In this appendix our goal is to find these distributions.

By defining the variables

$$z_{\pm}(x) = \frac{y'_{\pm}(x)}{y_{\pm}(x)}, \tag{B3}$$

we find from Eq. (B1) that $z_{\pm}(x)$ satisfy the stochastic Riccati equation

$$z'_{\pm}(x) = -z_{\pm}^2(x) - 2F(x)z_{\pm} + 2a_{\pm}. \tag{B4}$$

However, now the boundary conditions for $z_{\pm}$ in Eq. (25) are not specified. Therefore, for each realizations of $\{\xi(x)\}$, the solutions of $z_{\pm}(x)$ involve one unknown each that cannot be eliminated due to the lack of boundary conditions. In other words, to find the distributions of $z_{\pm}(x)$, we need the respective distributions at some initial points, which are unfortunately not specified.

It turns out, however, that this difficulty can be bypassed by a method [37,38,41] which lets us compute the distributions of $z_+(0)$ and $z_-(0)$ without having knowledge of the boundary conditions on $z_+(\infty)$ and $z_-(-\infty)$. We will present the method below for the present context.

First we consider Eq. (B4) for $x>0$—i.e.,

$$z'_+(x) = -z_+^2(x) - 2[\mu + \sqrt{\sigma}\xi(x)]z_+(x) + 2a_+. \tag{B5}$$

Note that $z_+(x)=y'_+(x)/y_+(x)$ is negative. We make a change of variable $x=-\tau$ and substitute $z_+(-\tau)=-\exp[\phi(\tau)]$ in Eq. (B5) to find that the new variable $\phi(\tau)$ satisfies a much simpler stochastic differential equation

$$\frac{d\phi}{d\tau} = b(\phi) + 2\sqrt{\sigma}\tilde{\xi}(\tau), \tag{B6}$$

where $\tilde{\xi}(\tau)=\xi(-\tau)$ and thus $\langle\tilde{\xi}(\tau)\rangle=0$ and $\langle\tilde{\xi}(\tau)\tilde{\xi}(\tau')\rangle=\delta(\tau-\tau')$. The source term $b(\phi)$ is given by

$$b(\phi) = -e^{\phi} + 2a_+e^{-\phi} + 2\mu. \tag{B7}$$

Now we can interpret Eq. (B6) as a simple Langevin equation describing the evolution of a Brownian particle starting at time $\tau=-\infty$ in a classical stable potential $U_{\rm cl}(\phi)=-\int_0^{\phi}b(\varphi)d\varphi=e^{\phi}+2a_+e^{-\phi}-2\mu\phi-(2a_++1)$. Even though we do not know the starting position of the particle $\phi(-\infty)$, it is completely irrelevant. No matter what the initial position is, eventually after a long time—i.e., when τ is far away from $-\infty$—the system will reach equilibrium and hence the stationary probability distribution of ϕ is simply given by the Gibbs measure

$$P_{\rm st}(\phi) = A\exp\left[-\frac{1}{2\sigma}U_{\rm cl}(\phi)\right] = A\exp\left[\frac{1}{2\sigma}\int_0^{\phi}b(\varphi)d\varphi\right], \tag{B8}$$

where A is a normalization constant such that $\int_{-\infty}^{\infty}P_{\rm st}(\phi)d\phi=1$. Now changing back to the original variable $z_+(x)$ we obtain the distribution of $z_+(0)$ as

$$P^+(-z_+(0)=w) = \frac{1}{\Omega_+}w^{\mu/\sigma-1}\exp\left[-\frac{1}{2\sigma}\left\{w+\frac{2a_+}{w}\right\}\right], \tag{B9}$$

where

$$\Omega_+ = \int_0^{\infty} w^{\mu/\sigma-1}\exp\left[-\frac{1}{2\sigma}\left\{w+\frac{2a_+}{w}\right\}\right]dw = 2(2a_+)^{\mu/2\sigma}K_{\mu/\sigma}\left(\frac{\sqrt{2a_+}}{\sigma}\right). \tag{B10}$$

Similarly for $x<0$, by putting $F(x)=-\mu+\sqrt{\sigma}\xi(x)$ in Eq. (B4) and substituting $z_-(x)=\exp[\phi(x)]$ one finds that $\phi(x)$ satisfies the same differential equation in x as Eq. (B6) with $\tilde{\xi}(x)=-\xi(x)$ and a_+ is replaced with a_-. Therefore $\phi(x)$ has the same stationary distribution as Eq. (B8) and consequently the distribution of $z_-(0)$ is same as that of $-z_+(0)$: namely,

$$P^-(z_-(0)=w) = \frac{1}{\Omega_-}w^{\mu/\sigma-1}\exp\left[-\frac{1}{2\sigma}\left\{w+\frac{2a_-}{w}\right\}\right], \tag{B11}$$

with

$$\Omega_- = \int_0^{\infty} w^{\mu/\sigma-1}\exp\left[-\frac{1}{2\sigma}\left\{w+\frac{2a_-}{w}\right\}\right]dw = 2(2a_-)^{\mu/2\sigma}K_{\mu/\sigma}\left(\frac{\sqrt{2a_-}}{\sigma}\right). \tag{B12}$$

Note that the distributions of $-z_+(0)$ and $z_-(0)$ given by Eqs. (B9) and (B11) have maxima at $(\mu-\sigma)+\sqrt{(\mu-\sigma)^2+2a_{\pm}}$, respectively, and in the limit $\sigma\to 0$ the distributions tend to delta functions around their maxima. Therefore in the limit $\sigma\to 0$ one recovers the pure case results by using the distributions $P^+(z_+(0))=\delta(z_+(0)+[\mu+\sqrt{\mu^2+2a_+}])$ and $P^-(z_-(0))=\delta(z_-(0)-[\mu+\sqrt{\mu^2+2a_-}])$.

APPENDIX C: LEFT HALF OF THE DISORDER-AVERAGED PDF OF THE OCCUPATION TIME FOR THE SINAI POTENTIAL (μ=0 AND σ>0)

By taking the disorder average of Eq. (114) one gets

$$\int_0^{\infty} dt e^{-\alpha t}\int_0^t dT e^{-pT}\overline{R_L(T|t)} = \frac{1}{\alpha}\overline{\ell_1(\alpha,p)}. \tag{C1}$$

Using the distributions of $-z_+(0)$ and $z_-(0)$ from Eqs. (B9) and (B11), respectively, with $a_+=\alpha+p$ and $a_-=\alpha$, from Eq. (109) one gets

$$\overline{\ell_1(\alpha,p)} = \frac{m_1(\alpha,p)}{\Omega_+\Omega_-}, \tag{C2}$$

where

$$m_1(\alpha,p) = \int_0^\infty \frac{dw_1}{w_1} \exp\left[-\frac{1}{2\sigma}\left(w_1 + \frac{2(\alpha+p)}{w_1}\right)\right] \times \int_0^\infty \frac{dw_2}{w_1+w_2} \exp\left[-\frac{1}{2\sigma}\left(w_2 + \frac{2\alpha}{w_2}\right)\right] \quad \text{(C3)}$$

and

$$\Omega_+ = 2K_0\left(\frac{\sqrt{2(\alpha+p)}}{\sigma}\right), \quad \Omega_- = 2K_0\left(\frac{\sqrt{2\alpha}}{\sigma}\right). \quad \text{(C4)}$$

Before we proceed further, let us take a detour to check the normalization condition of $\overline{R_L(T|t)}$. By putting $p=0$ in the above equations we get

$$\Omega_+ = \Omega_- = \Omega = 2K_0\left(\frac{\sqrt{2\alpha}}{\sigma}\right) \quad \text{(C5)}$$

and

$$m_1(\alpha,0) = \int_0^\infty \int_0^\infty \frac{dw_1}{w_1}\frac{dw_2}{w_2}\left[\frac{w_2}{w_1+w_2}\right] \times \exp\left\{-\frac{1}{2\sigma}\left[w_1 + \frac{2\alpha}{w_1}\right]\right\} \times \exp\left\{-\frac{1}{2\sigma}\left[w_2 + \frac{2\alpha}{w_2}\right]\right\}. \quad \text{(C6)}$$

Note that the above integral must remain invariant under the transformation $w_1 \leftrightarrow w_2$ of the dummy variables. Therefore we get

$$2m_1(\alpha,0) = \left[\int_0^\infty \frac{dw}{w} \exp\left\{-\frac{1}{2\sigma}\left[w + \frac{2\alpha}{w}\right]\right\}\right]^2 = \Omega^2. \quad \text{(C7)}$$

Therefore we have $\overline{\ell_1(\alpha,0)}=1/2$, and inverting the Laplace transform in Eq. (C1) with respect to α for $p=0$ gives the normalization condition $\int_0^t \overline{R_L(T|t)}dT=1/2$.

Now we analyze the large-t behavior of $\overline{R_0(T|t)}$. By making a change of variable $z=\alpha t$, it follows from Eq. (C1) that

$$\int_0^\infty dz e^{-z} \int_0^{z/\alpha} dT e^{-pT} \overline{R_0(T,z/\alpha)} = \overline{\ell_1(\alpha,p)}. \quad \text{(C8)}$$

In the $\alpha \to 0$ limit, $\Omega_+ = 2K_0(\sqrt{2p}/\sigma)$ and $\Omega_- \sim -\ln \alpha$. Therefore, from Eq. (C2),

$$\overline{\ell_1(\alpha \to 0,p)} = \frac{m_1(0,p)}{[-2K_0(\sqrt{2p}/\sigma)\ln \alpha]}, \quad \text{(C9)}$$

which suggest the following form for $\overline{R_L(T|t)}$ at large t:

$$\overline{R_L(T|t)} = \frac{1}{\ln t} R(T), \quad \text{(C10)}$$

where $R(T)$ is independent of t.

Now using Eqs. (C9) and (C10), in the limit $\alpha \to 0$, Eq. (C8) gives

$$\int_0^\infty dT e^{-pT} R(T) = \frac{m_1(0,p)}{2K_0(\sqrt{2p}/\sigma)}, \quad \text{(C11)}$$

where $m_1(0,p)$ is obtained from Eq. (C3),

$$m_1(0,p) = \int_0^\infty \frac{dw_1}{w_1} \exp\left[-\frac{1}{2\sigma}\left(w_1 + \frac{2p}{w_1}\right)\right] \times \int_0^\infty dw_2 \frac{e^{-w_2/2\sigma}}{w_1+w_2}. \quad \text{(C12)}$$

By making a change of variables $w_1=2\sigma x$ and $w_2=2\sigma y$ in the integrals in the above equation one gets

$$m_1(0,p) = \int_0^\infty \frac{dx}{x} \exp\left[-\left(x + \frac{p}{2\sigma^2 x}\right)\right] \int_0^\infty dy \frac{e^{-y}}{y+x}, \quad \text{(C13)}$$

where now the integral over y can be expressed in terms of the incomplete gamma function [44]:

$$\int_0^\infty dy \frac{e^{-y}}{y+x} = e^x \Gamma(0,x). \quad \text{(C14)}$$

Therefore, after straightforward simplification, Eq. (C13) becomes

$$m_1(0,p) = \int_0^\infty \frac{dx}{x} e^{-x} \Gamma(0,p/2\sigma^2 x). \quad \text{(C15)}$$

Now we will analyze the limiting behavior of $R(T)$ for small and large T by taking the limit of large and small p, respectively.

Since for large p

$$\Gamma(0,p/2\sigma^2 x) \approx \frac{2\sigma^2 x}{p} e^{-p/2\sigma^2 x}, \quad \text{(C16)}$$

from Eq. (C15), one gets

$$m_1(0,p) \approx \frac{2\sigma^2}{p}\int_0^\infty dx \exp\left[-\left(x + \frac{p}{2\sigma^2 x}\right)\right] = \frac{2\sqrt{2}\sigma}{\sqrt{p}} K_1\left(\frac{\sqrt{2p}}{\sigma}\right). \quad \text{(C17)}$$

Since the asymptotic behavior of $K_\nu(x)$ is independent of ν, substituting $m_1(0,p)$ from above in Eq. (C11) gives

$$\int_0^\infty dT e^{-pT} R(T) \approx \frac{\sqrt{2}\sigma}{\sqrt{p}}, \quad \text{(C18)}$$

for small p, and by inverting the Laplace transform with respect to p one obtains

$$R(T) \approx \frac{\sqrt{2}\sigma}{\sqrt{\pi T}}, \quad \text{as } T \to 0. \quad \text{(C19)}$$

To obtain the large-T behavior, we first consider the following integral:

$$\mathcal{D}(z) = \int_0^\infty \frac{dx}{x} e^{-xz} \Gamma(0, p/2\sigma^2 x), \tag{C20}$$

where $\mathcal{D}(1)=m_1(0,p)$ follows from Eq. (C15). Now by differentiating $\mathcal{D}(z)$ with respect to z, one can express it in terms of the modified Bessel function as [44]

$$\mathcal{D}'(z) = -\int_0^\infty dx e^{-xz} \Gamma(0, p/2\sigma^2 x) = -\frac{2}{z} K_0\left(\frac{\sqrt{2pz}}{\sigma}\right). \tag{C21}$$

Now by integrating back again with respect to z, we obtain $m_1(0,p)$ as

$$m_1(0,p) = \mathcal{D}(1) = 2\int_1^\infty \frac{dz}{z} K_0\left(\frac{\sqrt{2pz}}{\sigma}\right) = 4\int_{\sqrt{2p}/\sigma}^\infty \frac{dx}{x} K_0(x), \tag{C22}$$

where we have made the change of variable $2pz/\sigma^2 = x^2$. The $p \to 0$ limit can be obtained from the limiting behavior of the integral [45],

$$\int_y^\infty \frac{K_0(x)dx}{x} \sim \frac{1}{2}(\ln y)^2, \quad \text{as } y \to 0, \tag{C23}$$

which gives

$$m_1(0,p) \sim \frac{1}{2}(\ln p)^2, \quad \text{as } p \to 0. \tag{C24}$$

Since $K_0(\sqrt{2p}/\sigma) \sim -\frac{1}{2}\ln p$, as $p \to 0$, Eq. (C11) gives

$$\int_0^\infty dT e^{-pT} R(T) \sim -\frac{1}{2}\ln p, \tag{C25}$$

as $p \to 0$. Thus, inverting the Laplace transform with respect to p one obtains

$$R(T) \sim \frac{1}{2T}, \quad \text{as } T \to \infty. \tag{C26}$$

APPENDIX D: LEFT HALF OF THE DISORDER-AVERAGED PDF OF THE OCCUPATION TIME FOR $\mu > 0$

By taking the disorder average of Eq. (114) one gets

$$\int_0^\infty dt e^{-\alpha t} \int_0^t dT e^{-pT} \overline{R_L(T|t)} = \frac{1}{\alpha} \overline{\ell_1(\alpha,p)}. \tag{D1}$$

As in the pure case ($\sigma=0$), one also expects the $t \to \infty$ behavior of the disorder-averaged distribution to tend to a t-independent form

$$\lim_{t\to\infty} \overline{R_L(T|t)} = \overline{R_L(T)}. \tag{D2}$$

Therefore Eq. (D1) becomes

$$\int_0^\infty dT e^{-pT} \overline{R_L(T)} = \overline{\ell_1(0,p)}. \tag{D3}$$

Using the distributions of $-z_+(0)$ and $z_-(0)$ from Eqs. (B9) and (B11), respectively, with $a_+=p$ and $a_-=0$, from Eq. (109) one gets

$$\overline{\ell_1(0,p)} = \frac{1}{\Omega_+\Omega_-} \int_0^\infty dw_1 w_1^{\mu/\sigma-1} \exp\left[-\frac{1}{2\sigma}\left(w_1 + \frac{2p}{w_1}\right)\right] \times \int_0^\infty dw_2 \frac{w_2^{\mu/\sigma} e^{-w_2/2\sigma}}{w_1+w_2}, \tag{D4}$$

with

$$\Omega_+ = 2(2p)^{\mu/2\sigma} K_{\mu/\sigma}\left(\frac{\sqrt{2p}}{\sigma}\right),$$

$$\Omega_- = (2\sigma)^{\mu/\sigma} \Gamma(\mu/\sigma). \tag{D5}$$

Now the integral over w_2 in Eq. (D4) can be expressed as [44]

$$\int_0^\infty dw_2 \frac{w_2^{\mu/\sigma} e^{-w_2/2\sigma}}{w_1+w_2} = w_1^{\mu/\sigma} e^{w_1/2\sigma} \Gamma\left(\frac{\mu}{\sigma}+1\right) \Gamma\left(-\frac{\mu}{\sigma}, \frac{w_1}{2\sigma}\right), \tag{D6}$$

where $\Gamma(\alpha,x)$ is the incomplete gamma function. Therefore Eq. (D4) becomes

$$\overline{\ell_1(0,p)} = \frac{\mu m_2(p)}{(2\sigma)^{\mu/\sigma+1}(2p)^{\mu/2\sigma} K_{\mu/\sigma}(\sqrt{2p}/\sigma)}, \tag{D7}$$

where

$$m_2(p) = \int_0^\infty dw_1 w_1^{2\mu/\sigma-1} e^{-p/\sigma w_1} \Gamma\left(-\frac{\mu}{\sigma}, \frac{w_1}{2\sigma}\right). \tag{D8}$$

The small- and large-T behavior of $\overline{R_L(T)}$ can be found by analyzing Eq. (D7) in the limiting cases $p \to \infty$ and $p \to 0$, respectively.

Making a change of variable $w_1 = p/\sigma x$ in Eq. (D8) and then taking the $p \to \infty$ in the incomplete gamma function,

$$\Gamma\left(-\frac{\mu}{\sigma}, \frac{p}{2\sigma^2 x}\right) \sim \left(\frac{p}{2\sigma^2 x}\right)^{-\mu/\sigma-1} \exp\left(-\frac{p}{2\sigma^2 x}\right), \tag{D9}$$

gives

$$m_2(p) \approx (2\sigma)^2 (2p)^{\mu/\sigma-1} \int_0^\infty dx x^{-\mu/\sigma} \exp\left[-\left(x + \frac{p}{2\sigma^2 x}\right)\right], \tag{D10}$$

where the integral above on the right-hand side can further be expressed in terms of the modified Bessel function as [44]

$$\int_0^\infty dx x^{-\mu/\sigma} \exp\left[-\left(x+\frac{p}{2\sigma^2 x}\right)\right] = 2(2\sigma)^{\mu/\sigma-1}(2p)^{-\mu/2\sigma+1/2} K_{\mu/\sigma-1}\left(\frac{\sqrt{2p}}{\sigma}\right). \quad \text{(D11)}$$

Since the large-x behavior or $K_\nu(x)$ is independent of ν, Eq. (D7) simplifies to

$$\overline{\ell_1(0,p)} \approx \frac{\mu\sqrt{2}}{\sqrt{p}} \quad \text{as } p\to\infty. \quad \text{(D12)}$$

Therefore, by inverting the Laplace transform in Eq. (D3) with respect to p, one gets

$$\overline{R_L(T)} \approx \frac{\mu\sqrt{2}}{\sqrt{\pi T}}, \quad \text{for small } T. \quad \text{(D13)}$$

Now we will analyze the the large-T behavior by taking the limit $p\to 0$ in Eq. (D7). It is straightforward to obtain from Eq. (D8) that

$$m_2(0) = \frac{(2\sigma)^{2\mu/\sigma+1}\Gamma(\mu/\sigma)}{4\mu}, \quad \text{(D14)}$$

which gives

$$\overline{\ell_1(0,p)} \approx \frac{1}{4}\Gamma\left(\frac{\mu}{\sigma}\right)\left(\frac{\sigma\sqrt{2}}{\sqrt{p}}\right)^{\mu/\sigma} K^{-1}_{\mu/\sigma}\left(\frac{\sqrt{2p}}{\sigma}\right), \quad \text{(D15)}$$

for small p. However, if one takes the limit $p\to 0$ now in $K_{\mu/\sigma}(\sqrt{2p}/\sigma)$ in the above expression, it only gives the normalization condition $\int_0^\infty \overline{R_L(T)}dT=1/2$ and does not provide any information about the large-T behavior of $\overline{R_L(T)}$.

We make the ansatz

$$\overline{R_L(T)} \sim e^{-bT} \quad \text{(D16)}$$

for large T. Then the Laplace transform

$$\int_0^\infty dT e^{-pT}\overline{R_L(T)} \approx \frac{1}{p+b} \quad \text{(D17)}$$

for small p. Therefore substituting Eqs. (D15) and (D17) in Eq. (D3), one can conclude that b is given by the zero of $K_{\mu/\sigma}(\sqrt{2p}/\sigma)$ closest to origin in the left part of the complex-p plane.

APPENDIX E: DISORDER-AVERAGED PDF OF THE OCCUPATION TIME FOR $\mu<0$

Taking the disorder average of Eq. (108) gives

$$\int_0^\infty dt e^{-\alpha t}\int_0^t dT e^{-pT}\overline{P_{\mathrm{occ}}(T|t)} = \frac{1}{\alpha+p} + \frac{p}{\alpha(\alpha+p)}\overline{\ell_1(\alpha,p)}, \quad \text{(E1)}$$

where we have substituted $\overline{\ell_2(\alpha,p)}=1-\overline{\ell_1(\alpha,p)}$.

We are interested in finding the behavior of $\overline{P_{\mathrm{occ}}(T|t)}$, in the scaling limit $t\to\infty, T\to\infty$, but keeping $T/t=y$ fixed, which corresponds to the limit of conjugate variables: $\alpha\to 0, p\to 0$, keeping $p/\alpha=s$ fixed. Substituting $z=\alpha t$, $T=yz/\alpha$, and $p=\alpha s$ in Eq. (E1), we get

$$\int_0^1 dy\int_0^\infty dz e^{-(1+sy)z}\left[\frac{z}{\alpha}\overline{P_{\mathrm{occ}}\left(\frac{yz}{\alpha}\Big|\frac{z}{\alpha}\right)}\right] = \frac{1+sm_3(\alpha,s)}{(1+s)}, \quad \text{(E2)}$$

where $m_3(\alpha,s)=\overline{\ell_1(\alpha,\alpha s)}$. Equation (E2) suggests the form

$$\overline{P_{\mathrm{occ}}(T|t)} = \frac{1}{t}f_0(T/t) \quad \text{(E3)}$$

in the scaling limit $t\to\infty, T\to\infty$, while their ratio T/t is kept fixed. In the limit $\alpha\to 0$, by substituting the above scaling form in Eq. (E2), it is straightforward to obtain

$$\int_0^1 dy\frac{f_0(y)}{1+sy} = \frac{1+sm_3(0,s)}{(1+s)}, \quad \text{(E4)}$$

where putting $s=0$ gives the normalization condition $\int_0^1 f_0(y)dy=1$. Using this normalization, the above equation can be simplified to the following elegant form

$$\int_0^1 \frac{1-y}{1+sy}f_0(y)dy = m_3(0,s). \quad \text{(E5)}$$

Now by using the distributions of $z_\pm(0)$ from Eqs. (B9) and (B11) with $a_-=\alpha$ and $a_+=\alpha(1+s)$, from Eq. (109) we get

$$m_3(\alpha,s) = \overline{\ell_1(\alpha,\alpha s)} = \frac{1}{\Omega_+\Omega_-}\int_0^\infty dw_1\int_0^\infty dw_2\frac{w_2}{w_1+w_2} \times w_1^{-\nu-1}\exp\left[-\frac{1}{2\sigma}\left\{w_1+\frac{2\alpha(1+s)}{w_1}\right\}\right] \times w_2^{-\nu-1}\exp\left[-\frac{1}{2\sigma}\left\{w_2+\frac{2\alpha}{w_2}\right\}\right], \quad \text{(E6)}$$

where

$$\Omega_+ = 2(2\alpha)^{-\nu/2}(1+s)^{-\nu/2}K_\nu\left(\frac{\sqrt{2\alpha(1+s)}}{\sigma}\right), \quad \text{(E7)}$$

$$\Omega_- = 2(2\alpha)^{-\nu/2}K_\nu\left(\frac{\sqrt{2\alpha}}{\sigma}\right), \quad \text{(E8)}$$

with $\nu=|\mu|/\sigma$. Note that we simply cannot take the limit $\alpha\to 0$ in the integrals in Eq. (E6), as it diverges in that limit. However, it is possible to extract the divergent contribution outside the integrals which finally cancels exactly with the divergence of $\Omega_\pm$. This is done by making the change of variables

$$\frac{\alpha(1+s)}{\sigma w_1} = x, \quad \frac{\alpha}{\sigma w_2} = y, \quad \text{(E9)}$$

in the integral to get

$$m_3(\alpha,s) = \frac{\sigma^{2\nu}\alpha^{-2\nu}(1+s)^{-\nu}}{\Omega_+\Omega_-}\int_0^\infty dx \int_0^\infty dy \frac{x^\nu y^{\nu-1}}{x+(1+s)y} \times \exp\left[-\left\{x+\frac{\alpha(1+s)}{2\sigma^2 x}\right\}\right]\exp\left[-\left\{y+\frac{\alpha}{2\sigma^2 y}\right\}\right]. \tag{E10}$$

Now the limit $\alpha \to 0$ can be taken in the above equation, as, in this limit, $\Omega_+ \to \sigma^\nu\alpha^{-\nu}(1+s)^{-\nu}\Gamma(\nu)$ and $\Omega_- \to \sigma^\nu\alpha^{-\nu}\Gamma(\nu)$. Therefore from Eq. (E10) we get

$$m_3(0,s) = \frac{1}{\Gamma^2(\nu)}\int_0^\infty dy y^{\nu-1}e^{-y}\int_0^\infty \frac{x^\nu e^{-x}}{x+(1+s)y}dx. \tag{E11}$$

Now the integration over x can be expressed in terms of the incomplete gamma function [44]

$$\Gamma(\rho,\lambda) = \frac{e^{-t}t^\rho}{\Gamma(1-\rho)}\int_0^\infty \frac{x^{-\rho}e^{-x}}{x+\lambda}dx \quad [\mathrm{Re}\,\rho < 1, \lambda > 0], \tag{E12}$$

which gives

$$m_3(0,s) = \frac{\nu(1+s)^\nu}{\Gamma(\nu)}\int_0^\infty y^{2\nu-1}e^{sy}\Gamma(-\nu,(1+s)y)dy. \tag{E13}$$

The right-hand side, however, is one of the integral representation of the Gauss's hypergeometric function $F(\alpha,\beta;\gamma;z)$ [44], which gives

$$m_3(0,s) = \frac{1}{2}F(1,\nu;2\nu+1;-s). \tag{E14}$$

Now by using another integral representation [44]

$$F(\alpha,\beta;\gamma;-s) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 \frac{y^{\beta-1}(1-y)^{\gamma-\beta-1}}{(1+sy)^\alpha}dy, \tag{E15}$$

we get

$$m_3(0,s) = \frac{1}{B(\nu,\nu)}\int_0^1 \frac{1-y}{1+sy}[y(1-y)]^{\nu-1}dy, \tag{E16}$$

where $B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the beta function [44].

Now by comparing Eq. (E16) with Eq. (E5), one immediately gets

$$f_0(y) = \frac{1}{B(\nu,\nu)}[y(1-y)]^{\nu-1}, \quad 0 \le y \le 1. \tag{E17}$$

[1] M. Kac, Trans. Am. Math. Soc. **65**, 1 (1949).
[2] M. Kac, in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Berkeley, 1950*, edited by J. Neyman (University of California Press, Berkeley, 1951), pp. 189–215.
[3] M. Yor, *Some Aspects of Brownian Motion* (Birkhauser, Basel, 1992), pt. I.
[4] M. Yor, *Exponential Functionals of Brownian Motion and Related Processes* (Springer, Berlin, 2000).
[5] D. Dufresne, Scand Actuar. J. **1990**(1–2), 39.
[6] H. Geman and M. Yor, Math. Finance **3**, 349 (1993).
[7] A. Comtet, J. Desbois, and C. Texier, J. Phys. A **38**, R341 (2005), for a recent review.
[8] See S. N. Majumdar, Curr. Sci. **89**, 2076 (2005), for a recent review on Brownian functionals in physics and computer science.
[9] S. N. Majumdar and A. J. Bray, Phys. Rev. E **65**, 051112 (2002).
[10] R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[11] P. Lévy, Compos. Math. **7**, 283 (1939).
[12] D. A. Darling and M. Kac, Trans. Am. Math. Soc. **84**, 444 (1957).
[13] J. Lamperti, Trans. Am. Math. Soc. **88**, 380 (1958).
[14] K. Itô and H. P. McKean, *Diffusion Processes and their Sample Paths* (Springer, Berlin, 1974).
[15] See S. Watanabe, Proc. Symp. Pure Math. **57**, 157 (1995), for a recent review of the occupation time in the mathematics literature.
[16] See J. Pitman and M. Yor, Bernoulli **9**, 1 (2003), for a review.
[17] A. N. Borodin and P. Salminen, *Handbook of Brownian Motion—Facts and Formulae*, 2nd ed. (Birkhäuser, Basel, 2002).
[18] I. Dornic and C. Godrèche, J. Phys. A **31**, 5413 (1998).
[19] T. J. Newman and Z. Toroczkai, Phys. Rev. E **58**, R2685 (1998).
[20] A. Dhar and S. N. Majumdar, Phys. Rev. E **59**, 6413 (1999).
[21] G. D. Smedt, C. Godrèche, and J. M. Luck, J. Phys. A **34**, 1247 (2001).
[22] C. Godrèche and J. M. Luck, J. Stat. Phys. **104**, 489 (2001).
[23] S. N. Majumdar and D. S. Dean, Phys. Rev. E **66**, 041102 (2002).
[24] G. Margolin and E. Barkai, Phys. Rev. Lett. **94**, 080601 (2005).
[25] G. C. M. A. Ehrhardt, S. N. Majumdar, and A. J. Bray, Phys. Rev. E **69**, 016106 (2004).
[26] See S. N. Majumdar, Curr. Sci. **77**, 370 (1999), for a recent review on persistence.
[27] G. Bel and E. Barkai, Phys. Rev. Lett. **94**, 240602 (2005).
[28] E. Barkai, e-print cond-mat/0601143.
[29] O. Bénichou, M. Coppey, J. Klafter, M. Moreau, and G. Oshanin, J. Phys. A **38**, 7205 (2005).
[30] J. Desbois, J. Phys. A **35**, L673 (2002).
[31] A. Comtet, J. Desbois, and S. N. Majumdar, J. Phys. A **35**,

L687 (2002).

[32] J. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).

[33] J. P. Bouchaud, A. Comtet, A. Georges, and P. Le Doussal, Ann. Phys. (N.Y.) **201**, 285 (1990).

[34] J. W. Haus and K. W. Kehr, Phys. Rep. **150**, 263 (1987).

[35] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987), republished in **51**, 187 (2002).

[36] Ya. G. Sinai, Theor. Probab. Appl. **27**, 256 (1982), originally published in Teor. Veroyatn. Ee Primen. **27**, 247 (1982).

[37] A. Comtet and D. S. Dean, J. Phys. A **31**, 8595 (1998).

[38] S. N. Majumdar and A. Comtet, Phys. Rev. E **66**, 061105 (2002).

[39] D. S. Fisher, P. Le Doussal, and C. Monthus, Phys. Rev. Lett. **80**, 3539 (1998).

[40] F. Iglói and C. Monthus, Phys. Rep. **412**, 277 (2005).

[41] C. Aslangul, N. Pottier, and D. Saint-James, Physica A **164**, 52 (1990).

[42] Z. Shi, Panoramas Synth. **12**, 53 (2001).

[43] S. N. Majumdar and A. Comtet, Phys. Rev. Lett. **89**, 060601 (2002b).

[44] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. (Academic, New York, 2000).

[45] *Handbook of Mathematical Functions*, 9th ed., edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1970).

[46] J. Bertoin, *Lévy processes*, Vol. 121 of *Cambridge Tracts in Mathematics* (Cambridge University Press, Cambridge, England, 1996).

[47] A. Baldassarri, J. P. Bouchaud, I. Dornic, and C. Godrèche, Phys. Rev. E **59**, R20 (1999).

[48] I. Dornic, A. Lemaître, A. Baldassarri, and H. Chaté, J. Phys. A **33**, 7499 (2000).